# A NOTE ON FUNCTIONAL CLT FOR TRUNCATED SUMS

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Abstract. Let $\{X, X_i\}_{i\geq 1}$ be i.i.d. random variables with a symmetric continuous distribution and $\mathrm{E}X^2 = \infty$, and $\{b_n\}_{n\geq 1}$ be a sequence of increasing positive numbers. When $X$ belongs to the Feller class, and $n\mathrm{P}\big(|X| > b_n\big) \sim \gamma_n \uparrow \infty$, a functional CLT for the truncated sums $S_n = \sum_{i=1}^{n} X_i \mathrm{I}_{|X_i|\leq b_n}$ is proved.

## 1. Introduction

Let $\{X, X_i\}_{i\geq 1}$ be i.i.d. random variables with a symmetric continuous distribution and $\mathrm{E}X^2 = \infty$. Let $\{b_n\}_{n\geq 1}$ be a sequence of positive numbers such that $b_n \uparrow$ and

$$n\mathrm{P}\big(|X| > b_n\big) \sim \gamma_n \uparrow .$$

The *truncated sums* $S_n$ we will consider are defined by

$$S_n = \sum_{i=1}^{n} X_i \mathrm{I}_{|X_i|\leq b_n}. \tag{1}$$

**Remark 1.** For each fixed $n$ the truncated sum $S_n$ is a sum of independent identically distributed random variables bounded by $b_n$. If the random variable $X$ belongs to the Feller class, i.e.

$$\limsup_{t\to\infty} \frac{t^2\mathrm{P}(|X| > t)}{\mathrm{E}\big(X^2\mathrm{I}_{|X|\leq t}\big)} < \infty,$$

and the average number of the excluded terms

$$\gamma_n \uparrow +\infty$$

then

$$b_n^2 = o(B_n),$$

where $B_n = \mathrm{Var}(S_n)$. Hence, by applying the classical CLT theorem for independent random variables (for instance, Petrov (1995, p. 113)) we immediately get that $S_n/\sqrt{B_n}$ is asymptotically normal. But the functional version CLT for the truncated sums cannot be proved so easily. The main difficulty is that the sequence of the truncated sums $\{S_n\}$ *is not* a stochastic process with independent increments. However, as we will see the sequence of the truncated sums forms a martingale, so we can use the martingale analogs of the classical functional CLT to prove an invariance principle for $S_n$.

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**Remark 2.** The truncated sum can also be used for analysis of trimmed sums. Let $\{X_{k,n} : 0 \le k \le n\}$ be the order statistics of $|X_1|, |X_2|, ..., |X_n|$, that is

$$|X_{1,n}| \le ... \le |X_{n,n}|.$$

Let us consider the trimmed sum $T_n$ defined by

$$T_n = \sum_{i=1}^{n-k(n)} X_{i,n}, \tag{2}$$

where the number of the excluded terms $k(n) \sim \gamma_n$. It is quite intuitive that under certain conditions the truncated and trimmed sums will be very close (see Hahn, Kuelbs, and Weiner (1991, p. 30)). Thus, the functional CLT for the truncated sums $S_n$ can be treated as a first step toward the functional CLT for the trimmed sum $T_n$ in the case of the "intermediate trimming". For related results see Pruitt (1988), Griffin and Pruitt (1987), Hahn and Kuelbs (1989), Griffin and Mason (1991), Ould-Rouis (1991), Whalen (1992), and Kasahara (1993).

## 2. Conditioning: Auxiliary results

Here $\xi$, $\eta$, and $\zeta$ are integrable random variables.

**Lemma 1.** *Let $\xi_A = \xi \mathrm{I}_{\xi\in A}$ and $\xi_B = \xi \mathrm{I}_{\xi\in B}$, where $A, B \in \mathcal{B}$, Borel $\sigma$-field, and $AB = \emptyset$. Then*

$$\mathrm{E}(\xi_A|\xi_B) = \frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B = 0)} \mathrm{I}_{\xi_B=0}.$$

*Proof.* Let $C \in \mathcal{B}$. Then we have

$$\begin{aligned} \mathrm{E}\big(\xi_A \mathrm{I}_{\xi_B\in C}\big) &= \mathrm{E}\big(\xi \mathrm{I}_{\xi\in A}\big[\mathrm{I}_{\xi_B\in C\cap\{0\}} + \mathrm{I}_{\xi_B\in C\setminus\{0\}}\big]\big) \\ &= \mathrm{E}\big(\xi \mathrm{I}_{\xi\in A}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\{0\}\in C}\big) + \mathrm{E}\big(\xi \mathrm{I}_{\xi\in A}\mathrm{I}_{\xi\in B\cap\{C\setminus\{0\}\}}\big) \end{aligned}$$

The second term is equal to zero because $AB = \emptyset$. Since $\{\xi \in A\} \subseteq \{\xi_B = 0\}$ and $\mathrm{I}_{\{0\}\in C}$ is not a random variable we get

$$\mathrm{E}\big(\xi_A \mathrm{I}_{\xi_B\in C}\big) = \mathrm{I}_{\{0\}\in C}\mathrm{E}\xi_A.$$

Now

$$\begin{aligned} \mathrm{E}\big(\frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B = 0)}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\xi_B\in C}\big) &= \frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B = 0)}\mathrm{E}\big(\mathrm{I}_{\xi_B=0}\mathrm{I}_{\{0\}\in C}\big) \\ &= \mathrm{I}_{\{0\}\in C}\mathrm{E}\xi_A. \end{aligned}$$

Since the sets $\{\xi_B \in C\}$ form a $\pi$-system that generates the $\sigma$-field $\sigma(\xi_B)$ this finishes the proof (for instance, see Billingsley (1995, p. 446)). □

**Lemma 2.** *Let $\xi, \eta$, and $\zeta$ be random variables. Suppose $(\xi, \eta)$ and $\zeta$ are independent. Then*

$$\mathrm{E}\big(\xi|\sigma(\eta, \zeta)\big) = \mathrm{E}\big(\xi|\eta\big).$$

*Proof.* Let $C_1$ and $C_2$ be Borel sets.

$$\begin{aligned}\mathrm{E}\big(\xi \mathrm{I}_{\eta\in C_1,\zeta\in C_2}\big) &= \mathrm{E}\big(\xi \mathrm{I}_{\eta\in C_1}\mathrm{I}_{\zeta\in C_2}\big)\\ &= \mathrm{E}\big(\xi \mathrm{I}_{\eta\in C_1}\big)\mathrm{E}\big(\mathrm{I}_{\zeta\in C_2}\big)\\ &= \mathrm{E}\big(\mathrm{E}\big(\xi|\eta\big)\mathrm{I}_{\eta\in C_1}\big)\mathrm{E}\big(\mathrm{I}_{\zeta\in C_2}\big)\\ &= \mathrm{E}\big(\mathrm{E}\big(\xi|\eta\big)\mathrm{I}_{\eta\in C_1}\mathrm{I}_{\zeta\in C_2}\big)\\ &= \mathrm{E}\big(\mathrm{E}\big(\xi|\eta\big)\mathrm{I}_{\eta\in C_1,\zeta\in C_2}\big)\end{aligned}$$

□

**Lemma 3.** *Let $\xi$ and $\eta$ be independent random variables, and $A$ and $B$ be Borel sets such that $AB = \emptyset$. Then*

$$\mathrm{E}(\xi_A\eta_A|\sigma(\xi_B,\eta_B)) = \frac{\mathrm{E}\xi_A\mathrm{E}\eta_A}{\mathrm{P}(\xi_B=0)\mathrm{P}(\eta_B=0)}\mathrm{I}_{\eta_B=0}\mathrm{I}_{\xi_B=0}.$$

*Proof.* Let $C_1$ and $C_2$ be Borel sets. By independence we have

$$\mathrm{E}\big(\xi_A\eta_A\mathrm{I}_{\xi_B\in C_1,\eta_B\in C_2}\big) = \mathrm{E}\big(\xi_A\mathrm{I}_{\xi_B\in C_1}\big)\mathrm{E}\big(\eta_A\mathrm{I}_{\eta_B\in C_2}\big).$$

Lemma 1 tells us that

$$\mathrm{E}\big(\xi_A\mathrm{I}_{\xi_B\in C_1}\big) = \mathrm{E}\left[\frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B=0)}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\xi_B\in C_1}\right]$$

and

$$\mathrm{E}\big(\eta_A\mathrm{I}_{\eta_B\in C_2}\big) = \mathrm{E}\left[\frac{\mathrm{E}\eta_A}{\mathrm{P}(\eta_B=0)}\mathrm{I}_{\eta_B=0}\mathrm{I}_{\eta_B\in C_2}\right].$$

Hence,

$$\begin{aligned}\mathrm{E}\big(\xi_A\eta_A\mathrm{I}_{\xi_B\in C_1,\eta_B\in C_2}\big) &= \mathrm{E}\left[\frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B=0)}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\xi_B\in C_1}\right]\mathrm{E}\left[\frac{\mathrm{E}\eta_A}{\mathrm{P}(\eta_B=0)}\mathrm{I}_{\eta_B=0}\mathrm{I}_{\eta_B\in C_2}\right]\\ &= \mathrm{E}\left[\frac{\mathrm{E}\xi_A}{\mathrm{P}(\xi_B=0)}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\xi_B\in C_1}\frac{\mathrm{E}\eta_A}{\mathrm{P}(\eta_B=0)}\mathrm{I}_{\eta_B=0}\mathrm{I}_{\eta_B\in C_2}\right]\\ &= \mathrm{E}\left[\frac{\mathrm{E}\xi_A\mathrm{E}\eta_A}{\mathrm{P}(\xi_B=0)\mathrm{P}(\eta_B=0)}\mathrm{I}_{\xi_B=0}\mathrm{I}_{\eta_B=0}\mathrm{I}_{\xi_B\in C_1,\eta_B\in C_2}\right]\end{aligned}$$

□

## 3. Martingale property of $S_n$

**Proposition 1.** *Let us define a $\sigma$-field $\mathcal{F}_n = \sigma\big(X_1\mathrm{I}_{|X_1|\le b_n},...,X_n\mathrm{I}_{|X_n|\le b_n}\big)$. Then the sequence $\{S_n,\mathcal{F}_n\}_{n\ge0}$ with $S_0 = 0$ and $\mathcal{F}_0 = \{\Omega,\emptyset\}$ is a martingale.*

*Proof.* It is obvious that $\{\mathcal{F}_n\}$ is a filtration, and $S_n$ is $\mathcal{F}_n$-measurable. We need to show that

$$\mathrm{E}\big(S_{n+1}-S_n|\mathcal{F}_n\big) = 0.$$

First we note that

$$S_{n+1}-S_n = X_{n+1}\mathrm{I}_{|X_{n+1}|\le b_{n+1}} + \sum_{i=1}^{n} X_i\mathrm{I}_{b_n<|X_i|\le b_{n+1}}$$

Because of independence and symmetry we have that

$$\mathrm{E}\big(X_{n+1}\mathrm{I}_{|X_{n+1}|\le b_{n+1}}|\mathcal{F}_n\big) = \mathrm{E}\big(X_{n+1}\mathrm{I}_{|X_{n+1}|\le b_{n+1}}\big) = 0.$$

By Lemmas 1 and 2 we get

$$\mathrm{E}\big(X_i \mathrm{I}_{b_n<|X_i|\le b_{n+1}}|\mathcal{F}_n\big) = \frac{\mathrm{E}\big(X_i \mathrm{I}_{b_n<|X_i|\le b_{n+1}}\big)}{\mathrm{P}\big(|X_i|>b_n\big)} \mathrm{I}_{|X_i|>b_n}.$$

Because of the symmetry of the distribution we also have

$$\mathrm{E}\big(X_i \mathrm{I}_{b_n<|X_i|\le b_{n+1}}\big) = 0.$$

therefore, $\mathrm{E}\big(S_{n+1}-S_n|\mathcal{F}_n\big)=0$. □

**Remark 3.** What is an appropriate normalization sequence for the truncated sums $S_n$? If we are interested only in the weak convergence of the truncated sums then the variance

$$B_n = \mathrm{E}(S_n)^2 = n\mathrm{E}\big(X^2 \mathrm{I}_{|X|\le b_n}\big) \tag{3}$$

is a good choice. However, if we want to know the behavior of truncated sums as a random process the predictable quadratic variation

$$\langle S\rangle_n = \sum_{i=1}^{n} \mathrm{E}\big(\big(S_i - S_{i-1}\big)^2|\mathcal{F}_{i-1}\big), \quad S_0 = 0, \quad \mathcal{F}_0 = \{\Omega, \emptyset\} \tag{4}$$

plays an important role (see Shiryaev (1995, p. 483), and it will be probably a better choice of a normalizing sequence because $S_n$ is a martingale. Note also that $\mathrm{E}\langle S\rangle_n = B_n$.

**Lemma 4.** *The predictable quadratic variation of $S_n$ is given by*

$$\langle S\rangle_n = \sum_{i=1}^{n} \mathrm{E}\big(X^2 \mathrm{I}_{|X|\le b_i}\big) + \sum_{i=2}^{n} \frac{\mathrm{E}\big(X^2 \mathrm{I}_{b_{i-1}<|X|\le b_i}\big)}{\mathrm{P}\big(|X|>b_{i-1}\big)} \sum_{j=1}^{i-1} \mathrm{I}_{|X_j|>b_{i-1}} \tag{5}$$

*Proof.* For $i>1$ we have

$$S_i - S_{i-1} = X_i \mathrm{I}_{|X_i|\le b_i} + \sum_{j=1}^{i-1} X_j \mathrm{I}_{b_{i-1}<|X_j|\le b_i}.$$

Hence, by Lemmas 1, 2, and 3 we get (note that the expected value of cross-terms is zero)

$$\begin{aligned}\mathrm{E}\big(\big(S_i - S_{i-1}\big)^2|\mathcal{F}_{i-1}\big) &= \mathrm{E}\big(X_i^2 \mathrm{I}_{|X_i|\le b_i}\big) + \sum_{j=1}^{i-1} \frac{\mathrm{E}\big(X_j^2 \mathrm{I}_{b_{i-1}<|X_j|\le b_i}\big)}{\mathrm{P}\big(|X_j|>b_{i-1}\big)} \mathrm{I}_{|X_j|>b_{i-1}} \\ &= \mathrm{E}\big(X^2 \mathrm{I}_{|X|\le b_i}\big) + \frac{\mathrm{E}\big(X^2 \mathrm{I}_{b_{i-1}<|X|\le b_i}\big)}{\mathrm{P}\big(|X|>b_{i-1}\big)} \sum_{j=1}^{i-1} \mathrm{I}_{|X_j|>b_{i-1}}\end{aligned}$$

□

## 4. Functional Central Limit Theorem for $S_n$

Let us define $S_n(t)$ be a random element of $\mathcal{C}[0,1]$ obtained by linear interpolation between the points $(0,0)$, $(B_1/B_n, S_1/\sqrt{B_n})$,..., $(1, S_n/\sqrt{B_n})$. More specifically,

$$S_n(t) = \frac{1}{\sqrt{B_n}}\left[S_i + \frac{tB_n - B_i}{(B_{i+1}-B_i)}(S_{i+1}-S_i)\right], \quad \frac{B_i}{B_n} \le t < \frac{B_{i+1}}{B_n}.$$

According to Brown (1971) (see also Hall and Heyde (1980, p. 99-100)) in order to establish the functional central limit theorem for truncated sum $S_n$ we need to verify the Lindeberg condition

$$\text{for all } \epsilon > 0, \quad \frac{1}{B_n}\sum_{i=1}^{n} \mathrm{E}\big((S_i - S_{i-1})^2 \mathrm{I}_{|S_i - S_{i-1}|>\epsilon\sqrt{B_n}}\big) \to 0, \quad \text{as } n \to \infty \tag{6}$$

and the weak law of large numbers for the predictable quadratic variation $\langle S\rangle_n$

$$\frac{\langle S\rangle_n}{B_n} \xrightarrow{\mathrm{P}} 1. \tag{7}$$

If conditions (6) and (7) hold, then

$$S_n(t) \xrightarrow{d} W \tag{8}$$

in the sense $\mathcal{C}[0,1]$ with uniform metric $\rho$ where $W$ is standard Brownian motion on $[0,1]$. In this case for any continuous functional $h : \mathcal{C}[0,1] \to \mathbb{R}$, we have $h(S_n(t)) \xrightarrow{d} h(W)$. In particular, if $h(x) = \sup_{t\in[0,1]} |x(t)|$, then

$$\frac{\max_{i\le n} |S_i|}{\sqrt{B_n}} \xrightarrow{d} \sup_{t\in[0,1]} |W(t)|.$$

The main result of this section is the following theorem.

**Theorem 1.** *If the random variable $X$ belongs to the Feller class*

$$\limsup_{t\to\infty} \frac{t^2 \mathrm{P}(|X| > t)}{\mathrm{E}\big(X^2 \mathrm{I}_{|X|\le t}\big)} < \infty, \tag{9}$$

*the average number of the excluded variables*

$$n\mathrm{P}\big(|X| > b_n\big) \sim \gamma_n \uparrow +\infty, \tag{10}$$

*and $B_n/B_{n+1} \to 1$ then $S_n(t) \xrightarrow{d} W$ in the sense $(\mathcal{C}[0,1], \rho)$.*

This theorem is an immediate implication of the following two lemmas.

**Lemma 5.** *If the conditions of Theorem 1 hold, then*

$$\mathrm{E}\left(\frac{\langle S\rangle_n - B_n}{B_n}\right)^2 \to 0. \tag{11}$$

*Proof.* If we introduce the notation

$$\alpha_i = \frac{\mathrm{E}\big(X^2 \mathrm{I}_{b_i<|X|\le b_{i+1}}\big)}{\mathrm{P}(|X| > b_i)},$$

then the predictable quadratic variation is given by

$$\begin{aligned}\langle S\rangle_n =& \mathrm{E}\big(X_1^2 \mathrm{I}_{|X_1|\le b_1}\big) + \mathrm{E}\big(X_2^2 \mathrm{I}_{|X_2|\le b_2}\big) + ... + \mathrm{E}\big(X_n^2 \mathrm{I}_{|X_n|\le b_n}\big)\\ &+ \alpha_1 \mathrm{I}_{|X_1|>b_1}\\ &+ \alpha_2 \mathrm{I}_{|X_1|>b_2} + \alpha_2 \mathrm{I}_{|X_2|>b_2}\\ &+ ...\\ &+ \alpha_{n-1} \mathrm{I}_{|X_1|>b_{n-1}} + \alpha_{n-1} \mathrm{I}_{|X_2|>b_{n-1}} + ... + 0\end{aligned}$$

Let us define random variable $Y_i^n$ as follows

$$Y_i^n = \alpha_i \mathrm{I}_{|X_i|>b_i} + ... + \alpha_{n-1} \mathrm{I}_{|X_i|>b_{n-1}}.$$

It easy to see that we also have

$$\begin{aligned} Y_i^n =& \alpha_i \mathrm{I}_{b_i<|X_i|\leq b_{i+1}} \\ &+ (\alpha_i + \alpha_{i+1})\mathrm{I}_{b_{i+1}<|X_i|\leq b_{i+2}} \\ &+ ... \\ &+ (\alpha_i + ... + \alpha_{n-1})\mathrm{I}_{b_{n-1}<|X_i|}. \end{aligned}$$

Now note that

$$\mathrm{E}Y_i^n = \mathrm{E}(X^2 \mathrm{I}_{b_i<|X|\leq b_n})$$

and

$$Y_i^n \leq \alpha_i + ... + \alpha_{n-1} \quad a.s.$$

Therefore, we get

$$\mathrm{Var}(Y_i^n) \leq \mathrm{E}[Y_i^n]^2 \leq (\alpha_i + ... + \alpha_{n-1})\mathrm{E}Y_i^n.$$

Since,

$$\alpha_i = \frac{\mathrm{E}\big(X^2 \mathrm{I}_{b_i<|X|\leq b_{i+1}}\big)}{\mathrm{P}(|X| > b_i)} = \frac{\mathrm{E}Y_i^n - \mathrm{E}Y_{i+1}^n}{\mathrm{P}(|X| > b_i)} \leq \frac{\mathrm{E}Y_i^n - \mathrm{E}Y_{i+1}^n}{\mathrm{P}(|X| > b_n)},$$

and, therefore,

$$\alpha_i + ... + \alpha_{n-1} \leq \frac{\mathrm{E}Y_i^n}{\mathrm{P}(|X| > b_n)},$$

we find that

$$\mathrm{Var}(Y_i^n) \leq \frac{[\mathrm{E}Y_i^n]^2}{\mathrm{P}(|X| > b_n)} \leq \frac{[\mathrm{E}Y_1^n]^2}{\mathrm{P}(|X| > b_n)} \leq \frac{B_n^2}{n^2 \mathrm{P}(|X| > b_n)} \sim \frac{B_n^2}{n\gamma_n}.$$

Thus, finally we get

$$\mathrm{E}\left(\frac{\langle S\rangle_n - B_n}{B_n}\right)^2 = \frac{\mathrm{Var}(\sum_{i=1}^{n-1} Y_i^n)}{B_n^2} \leq \frac{1}{B_n^2}\frac{B_n^2}{\gamma_n} = \frac{1}{\gamma_n} \to 0.$$

□

**Lemma 6.** *If the conditions of Theorem 1 hold, then the truncated sums $S_n$ satisfy the Lindenberg condition (6).*

*Proof.* By the Cauchy inequality we have

$$\mathrm{E}\big((S_i - S_{i-1})^2 \mathrm{I}_{|S_i - S_{i-1}|>\epsilon\sqrt{B_n}}\big) \leq \big(\mathrm{E}(S_i - S_{i-1})^4\big)^{1/2}\big(\mathrm{P}\big(|S_i - S_{i-1}| > \epsilon\sqrt{B_n}\big)\big)^{1/2}.$$

First note that by the Chebyshev inequality we have

$$\mathrm{P}\big(|S_i - S_{i-1}| > \epsilon\sqrt{B_n}\big) \leq \frac{B_i - B_{i-1}}{\epsilon^2 B_n}.$$

Now let us estimate $\mathrm{E}(S_i - S_{i-1})^4$. The martingale difference $S_i - S_{i-1}$ is given by

$$S_i - S_{i-1} = \xi_1 + ... + \xi_{i-1} + \xi_i,$$

where $\xi_i = X_i \mathrm{I}_{|X_i| \le b_i}$ and $\xi_j = X_j \mathrm{I}_{b_{i-1} < |X_j| \le b_i}$ for $j = 1, 2, ..., i-1$. Because of symmetry of the distribution and independence we get

$$\begin{aligned}\mathrm{E}(S_i - S_{i-1})^4 &= \sum_{j=1}^{i} \mathrm{E}\xi_j^4 + 6 \sum_{1 \le j < k \le i} \mathrm{E}\xi_j^2 \mathrm{E}\xi_k^2 \\ &\le \sum_{j=1}^{i} \mathrm{E}\xi_j^4 + 3 \left[ \sum_{j=1}^{i} \mathrm{E}\xi_j^2 \right]^2\end{aligned}$$

Since

$$\xi_j^2 \le b_i^2 \le b_n^2 \text{ for all } 1 \le j \le i$$

we find that

$$\mathrm{E}(S_i - S_{i-1})^4 \le b_n^2 (B_i - B_{i-1}) + 3(B_i - B_{i-1})^2.$$

Hence, we have that

$$\begin{aligned}\mathrm{E}\big((S_i - S_{i-1})^2 \mathrm{I}_{|S_i - S_{i-1}| > \epsilon\sqrt{B_n}}\big) &\le \frac{\sqrt{b_n^2 + 3(B_i - B_{i-1})}(B_i - B_{i-1})}{\epsilon\sqrt{B_n}} \\ &\le \frac{b_n(B_i - B_{i-1})}{\epsilon\sqrt{B_n}} + \frac{2(B_i - B_{i-1})^{3/2}}{\epsilon\sqrt{B_n}}.\end{aligned}$$

Thus, we get that

$$\frac{1}{B_n} \sum_{i=1}^{n} \mathrm{E}\big((S_i - S_{i-1})^2 \mathrm{I}_{|S_i - S_{i-1}| > \epsilon\sqrt{B_n}}\big) \le$$

$$\begin{aligned}&\le \frac{1}{B_n} \sum_{i=1}^{n} \frac{b_n(B_i - B_{i-1})}{\epsilon\sqrt{B_n}} + \frac{2}{B_n} \sum_{i=1}^{n} \frac{(B_i - B_{i-1})^{3/2}}{\epsilon\sqrt{B_n}} \\ &\le \frac{b_n}{\epsilon B_n^{3/2}} \sum_{i=1}^{n} (B_i - B_{i-1}) + \frac{2}{\epsilon B_n^{3/2}} \sum_{i=1}^{n} (B_i - B_{i-1})^{3/2} \\ &\le \frac{b_n}{\epsilon B_n^{1/2}} + \frac{2}{\epsilon B_n^{3/2}} \sum_{i=1}^{n} (B_i - B_{i-1})^{3/2}.\end{aligned}$$

Because of (9) and (10) we have

$$\frac{b_n}{\sqrt{B_n}} = O\left( \frac{1}{\sqrt{n\mathrm{P}(|X| > b_n)}} \right) \to 0$$

as $n \to \infty$, so the first term goes to zero. Finally, it is easy to show that

$$\frac{1}{B_n^{3/2}} \sum_{i=1}^{n} (B_i - B_{i-1})^{3/2} \le \left( \frac{\max_{i \le n} B_i - B_{i-1}}{B_n} \right)^{1/2} \to 0,$$

if $B_n / B_{n+1} \to 1$.

□

## 5. Concluding remarks

It is well known fact that $B_n$ is asymptotically equivalent to the variance of the trimmed sums $T_n$ defined by (2). So if we show that $S_n$ and $T_n$ are close in some appropriate sense, then we can substitute the truncated sums $S_n$ in Theorem 1 by the corresponding trimmed sum $T_n$. More specifically, one need to show that

$$\frac{\max\{|S_1 - T_1|, |S_2 - T_2|, ..., |S_n - T_n|\}}{\sqrt{B_n}} \xrightarrow{\text{P}} 0.$$

A similar result was proved in Egorov and Pozdnyakov (1997), where we showed that in case of the symmetric $X$ that belongs to the Feller class

$$\limsup \frac{|S_n - T_n|}{\sqrt{B_n \log\log B_n}} \to 0 \quad a.s.,$$

whenever $k(n)/\log\log n \to \infty$. However, it is still a bit less than we need for a functional CLT.

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