On the Martingale Framework for Futures Prices

Vladimir Pozdnyakov[1], **J. Michael Steele**

Department of Statistics, University of Connecticut, 215 Glenbrook Rd, Storrs CT 06269-4120, (e-mail: boba@stat.uconn.edu), (phone: 1-860-486-6979), (fax: 1-860-486-4113)
Department of Statistics, Wharton School, University of Pennsylvania, Huntsman Hall 447, University of Pennsylvania, Philadelphia PA 19104, (e-mail: steele@wharton.upenn.edu), (phone: 1-215-898-9477), (fax: 1-215-898-1280)

Abstract. We provide a framework for the martingale representation for futures prices which has some concrete advantages over the classical treatments of Duffie (2001) or Karatzas and Shreve (1998). In particular, the new formulation accommodates models where the distribution of the associated risk-free rate has unbounded support. This relaxation is particularly useful in the theory of LIBOR futures.

Key words: Futures prices, interest rates, LIBOR futures prices, arbitrage pricing, equivalent martingale measures, Heath-Jarrow-Morton models

[1]Corresponding author

1 Introduction

If $\phi(t)$ denotes the futures price at time t for a commodity, a stock index, or an interest rate index, then, in the context of the theory of complete financial markets, $\phi(t)$ is often represented by an identity of the form

$$\phi(t) = \widetilde{E}(B|\mathcal{F}_t), \tag{1}$$

where, in the typical case, B is simply the spot price at time T of the underlying asset. Here, of course, the indicated conditional expectation is taken with respect to the so-called risk-neutral probability measure $\widetilde{P}$, and $\mathcal{F}_t$ denotes the σ-field which ones uses to indicate the ensemble of information which is assumed to be available at time t.

Such a martingale representation is known to hold in those models where the instantaneous risk-free interest rate $r(\cdot)$ is technically well behaved, and, for example, Duffie (2001, p. 172) shows that martingale representation (1) holds if

$$P\left(A_1 \leq \inf_{0\leq t\leq T} r(t) \leq \sup_{0\leq t\leq T} r(t) \leq A_2\right) = 1 \tag{2}$$

for some constants $-\infty < A_1 \leq A_2 < \infty$. Karatzas and Shreve (1998, p. 45) also show that one has a martingale representation for $\phi(t)$ under the slightly weaker assumption that the accumulation factor

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right) \tag{3}$$

satisfies the almost sure boundedness condition

$$P\left(\alpha_1 \leq \inf_{0\leq t\leq T} \beta(t) \leq \sup_{0\leq t\leq T} \beta(t) \leq \alpha_2\right) = 1 \tag{4}$$

for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$.

Unfortunately, there are natural — and almost unavoidable — circumstances where both of these conditions fail to be met. For example, in any model that leads to a marginal Gaussian distribution for the risk-free rate $r(t)$, the spot rate boundedness condition (2) of Duffie will fail. Moreover, one can check without difficulty that the accumulation factor condition (4) of Karatzas and Shreve also fails under some simple cases, such as the Ho-Lee model where under the risk-neutral measure $\tilde{P}$ the risk-free rate $r(t)$ has a representation a $g(t) + \sigma\tilde{B}_t$ for a deterministic function g and a $\tilde{P}$-Brownian motion $\tilde{B}_t$.

To be sure, the Ho-Lee model is no longer anyone's first choice as a model for the risk-free rate, but analogous difficulties emerge with essentially all Gaussian term structure models, and sadly enough, it difficult to specify *any* feasible non-deterministic model for the risk-free rate where the conditions (2) or (4) will apply, even though parts of the conditions are easily met. For example, the lower bound on the deflator which one needs in the Karatzas-Shreve condition (4) is easy to satisfy; one just needs a model with $r(t) \geq 0$ for all $t \in [0, T]$.

Nevertheless, it is a ticklish matter to provide a feasible model for the process $\{r(t) : t \in [0,T]\}$ which will guarantee that the accumulation factor $\beta(t)$ is bounded from above with probability one.

Fortunately, one can avoid these difficulties by a small technical modification of the usual specification of a futures price process. The modification yields a broadly applicable sufficient condition for the martingale representation (1), and the two-sided bound (4) of Karatzas and Shreve can be replaced with a simpler one-sided condition that is more easily met. Despite the technical nature of the proposed changes, they deserve to be made. As we detail below, they rescue the theory of interest rate futures from the horns of a modelling dilemma.

2 The Martingale Representation of Futures Process

Like Karatzas and Shreve (1998), we take a *cumulative income process* to be a semimartingale $\{\phi(t) : 0 \le t \le T\}$, and informally we view $\phi(t)$ as the net amount of money received by the holder of an interest rate futures contact during the time interval $[0,t]$. We then write $\phi(\cdot)$ in the usual semimartingale decomposition

$$\phi(t) = \phi(0) + \phi^{fv}(t) + \phi^{lm}(t), \qquad 0 \le t \le T, \tag{5}$$

where $\phi^{fv}(\cdot)$ is a càdlàg process with finite variation and where $\phi^{lm}(\cdot)$ is a $\widetilde{P}$-local martingale. Here, $\widetilde{P}$ continues to refer to the equivalent martingale measure of an underlying standard financial market model $\mathcal{M}$ (as defined in Karatzas and Shreve (1998, p. 17)), and the decomposition (5) is unique $\widetilde{P}$-*a.s.* provided that we take standardized initial values $\phi^{fv}(0) = 0$ and $\phi^{lm}(0) = 0$. To be completely precise, we should call ϕ a cumulative cash flow process associated with the market model $\mathcal{M}$.

In parallel with Karatzas and Shreve (1998, p. 18), we say that the cumulative income process $\phi(\cdot)$ associated with a market model $\mathcal{M}$ is *integrable* provided that it satisfies the two integrability conditions

$$\widetilde{E}\int_0^T \beta(t)^{-1} d\widehat{\phi}^{fv}(t) < \infty \quad \text{and} \quad \widetilde{E}\int_0^T \beta(t)^{-2}\, d\langle \phi^{lm}\rangle(s) < \infty, \tag{6}$$

where $\widehat{\phi}^{fv}(t)$ denotes the total variation of $\phi^{fv}(\cdot)$ on $[0,t]$ and $\langle\phi^{lm}\rangle(t)$ denotes the quadratic variation of $\phi^{lm}(\cdot)$ on $[0,t]$. Such an integrable cumulative income process $\phi(\cdot)$ is called a *European contingent claim associated with the market model* $\mathcal{M}$, and, by the general theory of arbitrage prices (say as put in Proposition 2.3 of Karatzas and Shreve (1998, p. 41)), we know that the unique arbitrage-free price at time $t \in [0,T]$ of the European contingent claim $\phi(\cdot)$ is given by the classic pricing formula

$$\beta(t)\widetilde{E}\left[\int_t^T \beta(s)^{-1}\, d\phi(s) \,\middle|\, \mathcal{F}_t\right]. \tag{7}$$

We can now state our proposed definition for the futures price process associated with a market model.

Definition 1 (Futures Price Process) *If $\{\phi(t) : 0 \leq t \leq T\}$ is a European contingent claim associated with the market model $\mathcal{M}$, then $\phi(\cdot)$ is called a* futures price process *with terminal value $B \in \mathcal{F}_T$ provided that $\phi(\cdot)$ has the three following properties:*
1. $\phi(T) = B$,
2. the arbitrage-free price of $\phi(\cdot)$, which is given by formula (7), is equal to zero $\widetilde{P}$-a.s. for all $t \in [0, T]$, and finally
3. the process $\phi(\cdot)$ satisfies the regularity condition

$$\widetilde{E}\langle\phi^{lm}\rangle(T) < \infty, \tag{8}$$

where $\langle\phi^{lm}\rangle(\cdot)$ denotes the quadratic variation of $\phi^{lm}(\cdot)$.

The first of these conditions just reflects the required terminal value of the futures price process, while the second condition carries almost all of the modelling responsibility. Specifically, condition (2) reflects the fundamental fact that one can enter into a futures contract at any time on either the long or short side with zero cost, so the arbitrage free price of the associated cash flow must also equal zero — or else one would have an arbitrage possibility.

The third condition plays a technical role, and this only point where the Definition 1 differs from earlier treatments. We intend to argue that the added condition (8) provides a genuinely more appropriate setting for futures price modelling, but first we need a representation theorem.

Theorem 1 (Representation of Futures Prices) *Let B be an $\mathcal{F}_T$-measurable random variable such that*

$$\widetilde{E}[B^2] < \infty.$$

If a futures price process $\{\phi(t) : 0 \leq t \leq T\}$ associated with an market model $\mathcal{M}$ has terminal value B, then the process $\phi(\cdot)$ is a $\widetilde{P}$-martingale on $[0, T]$, and we have the representation

$$\phi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad \textit{for all} \quad 0 \leq t \leq T. \tag{9}$$

Conversely, if the martingale $\psi(\cdot)$ defined by

$$\psi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad \textit{for} \quad 0 \leq t \leq T \tag{10}$$

satisfies the integrability condition

$$\widetilde{E}\left[\int_0^T \beta(t)^{-2} d\langle\psi\rangle(t)\right] < \infty, \tag{11}$$

where $\langle\psi\rangle(\cdot)$ denotes the quadratic variation of $\psi(\cdot)$, then the process $\psi(\cdot)$ is the unique futures price process for $\mathcal{M}$ in the sense of Definition 1.

Proof. We have $P(\beta(t) = 0) = 0$ for all $t \in [0, T]$, so, from the zero-price constraint in the definition of a futures price process, we see that ϕ satisfies

$$\widetilde{E}\left[\int_t^T \beta(u)^{-1}\, d\phi(u) \,|\, \mathcal{F}_t\right] = 0 \quad \widetilde{P}\text{-}a.s. \quad \text{for all } t \in [0, T]. \tag{12}$$

If we define a new process $I(\cdot)$ by the stochastic integral

$$I(t) = \int_0^t \beta(u)^{-1}\, d\phi(u) \qquad 0 \le t \le T,$$

then (12) tells us that for all $t \in [0, T]$ the process $I(\cdot)$ satisfies

$$\widetilde{E}[I(T) \,|\, \mathcal{F}_t] = \int_0^t \beta(u)^{-1}\, d\phi(u) + \widetilde{E}\left[\int_t^T \beta(u)^{-1}\, d\phi(u) \,|\, \mathcal{F}_t\right] = I(t),$$

and from this identity we see that $I(\cdot)$ is a $\widetilde{P}$-martingale. Now, by its construction, $I(\cdot)$ has the stochastic differential $dI(t) = \beta(t)^{-1} d\phi(t)$, and, if multiply this equation by $\beta(t)$ and integrate, we find that $\phi(\cdot)$ is given by

$$\phi(t) - \phi(0) = \int_0^t \beta(u)\, dI(u).$$

This formula implies that the process $\phi(\cdot)$ is a $\widetilde{P}$-local martingale, so, in its semimartingale decomposition we have $\phi^{fv}(\cdot) \equiv 0$ and $\phi^{lm}(\cdot) = \phi(\cdot) - \phi(0)$.

These relations provide the required link to our condition (8) on the quadratic variation of $\phi^{lm}(\cdot)$, since it is well-known (say from Karatzas and Shreve (1997, p. 38)) that a local martingale with an integrable quadratic variation must be an honest square-integrable martingale. Thus $\phi(\cdot)$ is a martingale, and the proof of the direct half of the theorem is complete.

To prove the converse we first note that the terminal condition is trivial, so it suffices to check that the process $\psi(\cdot)$ defined by (10) is a European contingent claim that satisfies the zero price condition of required by the definition of a futures price process. By Jensen's inequality and square integrability hypothesis on B we see that the process $\psi(\cdot)$ is a square integrable $\widetilde{P}$-martingale, so its canonical decomposition as a semimartingale is trivially given by $\psi^{fv}(\cdot) \equiv 0$ and $\psi^{lm}(\cdot) = \psi(\cdot) - \psi(0)$ with $\widetilde{E}\langle \phi^{lm} \rangle(T) < \infty$. By our key assumption (11) on $\psi(\cdot)$ we have the second of the two integrability conditions (6). For $\psi(\cdot)$ the first condition is vacuous, so $\psi(\cdot)$ is a European contingent claim that satisfies the regularity condition (8).

Now, if we define a new process $J(\cdot)$ by the stochastic integral

$$J(t) = \int_0^t \beta(u)^{-1}\, d\psi(u) \qquad t \in [0, T],$$

then, by our hypothesis (11) on the quadratic variation of $\psi(\cdot)$ and the "well known fact" used just a few lines ago, we see that the process $J(\cdot)$ is a square-integrable $\widetilde{P}$-martingale. From the martingale property of $J(\cdot)$, we trivially find

$$\beta(t)\widetilde{E}[J(T) - J(t) | \mathcal{F}_t] = 0, \quad \text{for all } t \in [0, T].$$

This is the zero-price condition for the European contingent claim $\psi(\cdot)$, and the uniqueness assertion is immediate from the first part of the theorem, so the proof of the converse is complete. $\square$

3 The Explicit Connection

Theorem 1 quickly yields a martingale representation theorem for futures price processes under conditions that liberalize those that have traditionally been imposed on the short rate process.

Theorem 2 *Let B be an $\mathcal{F}_T$-measurable random variable such that*

$$\widetilde{E}[B^2] < \infty.$$

If the accumulation factor $\beta(\cdot)$ of the market model $\mathcal{M}$ satisfies the one-sided boundedness condition

$$P\left(\alpha \leq \inf_{0 \leq t \leq T} \beta(t)\right) = 1 \tag{13}$$

for a constant $0 < \alpha \leq \infty$, then there exists a unique futures price process $\psi(\cdot)$ associated with the standard financial market and the terminal value B. Moreover, $\psi(\cdot)$ has the martingale representation

$$\psi(t) = \widetilde{E}[B|\mathcal{F}_t] \quad \text{for all} \qquad t \in [0, T]. \tag{14}$$

Proof. According to Theorem 1, the process $\psi(\cdot)$ defined by formula (14) must be an honest futures price process if it satisfies the integrability condition (11), and, by our hypothesis (13) on the accumulation factor, we find

$$\widetilde{E} \int_0^T \beta(s)^{-2} \, d\langle \psi \rangle(s) \leq \frac{1}{\alpha^2} \widetilde{E} \int_0^T d\langle \psi \rangle(s) = \frac{1}{\alpha^2} \widetilde{Var}(B) < \infty. \tag{15}$$

Thus, $\psi(\cdot)$ satisfies the bound (11), and, by the first part of Theorem 1, we see that $\psi(\cdot)$ is indeed the *unique* futures price process with terminal value B. $\square$

4 An Example: The Simplest LIBOR Futures

Theorems 1 and 2 are pertinent to *any* futures price process, but to appreciate their contribution one might specifically focus on interest rate futures. In this case no one can argue that a deterministic model for the underlying risk-free rate would be reasonable.

Here we will consider a model for LIBOR futures where the associated term structure is governed by the Ho-Lee model. Even though the Ho-Lee model is no longer at the cutting edge, it does have the benefit of requiring very little overhead, and models of more contemporary interest are amenable to similar analyses.

Let $\{P(t, T)\}$ denote a family of prices for zero-coupon bonds in accordance with the Ho-Lee model, say as specified in Heath et al. (1992, p. 90). Next let

$L_\lambda(t)$ denote the LIBOR quote at time t for a deposit of 360λ days (so $\lambda = 1/4$ corresponds to a 90-day term of deposit) and recall the bookkeeping relationship between zero-coupon bond prices and LIBOR quotes:

$$L_\lambda(t) = \frac{1}{\lambda}\left(\frac{1}{P(t, t+\lambda)} - 1\right). \tag{16}$$

The converse half of Theorem 1 tells us that the process $F_\lambda(t,T)$ defined by

$$F_\lambda(t,T) = \widetilde{E}\left[100(1 - L_\lambda(T)) \mid \mathcal{F}_t\right] \qquad 0 \le t \le T, \tag{17}$$

will be a futures price process with terminal value $B = 100(1 - L_\lambda(T))$ if we can check two basic conditions. First, we need to show

$$\widetilde{E}[B^2] < \infty, \tag{18}$$

and, more delicately, we need to show that the quadratic variation of the process $F_\lambda(t,T)$ defined by (17) satisfies the integrability condition

$$\widetilde{E}\int_0^T \beta(s)^{-2} d\langle F_\lambda(\cdot, T)\rangle(s) < \infty. \tag{19}$$

These tasks are addressed by the following proposition.

Proposition 1 *Conditions (18) and (19) both hold under the Ho-Lee model, and consequently the process $F_\lambda(t,T)$ given by (17) is a futures price process in the sense of Definition 1.*

Proof. From the bookkeeping identity for the LIBOR quotes (16) we see that to prove (18), it suffices to show that $\widetilde{E}\left(P(T, T+\lambda)^{-2}\right) < \infty$. A short calculation analogous to that of Heath et al. (1992, p. 91) confirms that

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\frac{\sigma^2}{2} Tt(T-t) - \sigma(T-t)\widetilde{B}_t\right]. \tag{20}$$

From this formula, we see that $P(T, T+\lambda)^{-2}$ is a product of a deterministic function and a random variable with a lognormal distribution under $\widetilde{P}$, so $P(T, T+\lambda)^{-2}$ has a finite $\widetilde{P}$-expectation, exactly as required.

To prove the quadratic variation bound (19), we first note (cf. Musiela and Rutkowski (1997, p. 373)) that we have

$$F_\lambda(t,T) = 100\left(\frac{1}{\lambda} + 1\right) - \left(\frac{100}{\lambda}\right)\frac{P(t,T)}{P(t, T+\lambda)} G(t, T, \lambda), \tag{21}$$

where $G(t,T,\lambda) = \exp\left[(\sigma^2/8)(T-t)(T-t+1/2)\right]$. With help form the bond price representation (20), this formula leads to an explicit formula for the quadratic variation of $F_\lambda(t,T)$. Specifically, if we use (20) to eliminate the bond prices from (21), we find

$$F_\lambda(t,T) = 100\left(\frac{1}{\lambda} + 1\right) - D(t,T,\lambda)\exp(\sigma\lambda\widetilde{B}_t),$$

where in the last summand we have

$$D(t,T,\lambda) = \frac{100}{\lambda}\frac{P(0,T)}{P(0,T+\lambda)}\mathrm{e}^{\sigma^2(T+\lambda)t(T+\lambda-t)/2-\sigma^2 Tt(T-t)/2}G(t,T,\lambda).$$

The immediate application of Itô's formula might seem natural here, but it would be needlessly messy; one does much better to note first that the definition (17) of $F_\lambda(t,T)$ tells us $F_\lambda(t,T)$ is a $\widetilde{P}$-martingale, so the drift term of $dF_\lambda(t,T)$ must be zero and hence Itô's formula must give us simply

$$dF_\lambda(t,T) = -\sigma\lambda D(t,T,\lambda)\exp[\sigma\lambda\widetilde{B}]\, d\widetilde{B}_t.$$

The quadratic variation $\langle F_\lambda(\cdot,T)\rangle(\cdot)$ of the process $F_\lambda(\cdot,T)$ therefore satisfies the SDE $d\langle F_\lambda(\cdot,T)\rangle(t) = \sigma^2\lambda^2 D(t,T,\lambda)^2\exp[2\sigma\lambda\widetilde{B}_t]\,dt$. In the Ho-Lee model, the spot rate is given by $r(t) = f(0,t)+\sigma^2 t^2/2+\sigma\widetilde{B}_t$ so we find

$$\beta(t) = \exp\left[\int_0^t (f(0,s)+\sigma^2 s^2/2+\sigma\widetilde{B}_s)\,ds\right],$$

so the critical integral (19) has the representation

$$\int_0^T \beta(t)^{-2} d\langle F_\lambda(\cdot,T)\rangle(t) = \int_0^T D_1(t,T,\lambda)\exp\left[-\int_0^t 2\sigma\widetilde{B}_s\,ds + 2\sigma\lambda\widetilde{B}_t\right]dt, \qquad (22)$$

where we have set

$$D_1(t,T,\lambda) = \sigma^2\lambda^2 D(t,T,\lambda)^2\exp\left[-\int_0^t (2f(0,s)+\sigma^2 s^2)\,ds\right].$$

By a short calculation with Itô's formula and Itô's isometry, we then find that

$$-\int_0^t 2\sigma\widetilde{B}_s\,ds + 2\sigma\lambda\widetilde{B}_t = 2\sigma\int_0^t (s-\lambda-t)\,dB_s \stackrel{d}{=} N[0, 4\sigma^2\{\lambda^3/3-(\lambda-t)^3/3\}],$$

and this tells us that the inside integrand of the critical integral (22) has the lognormal distribution. Thus, by the boundedness of the deterministic function $D_1(t,T,\lambda)$, one obtains the finiteness of the expectation of (22) after routine estimates. □

This proposition illustrates the relative ease with which Theorem 1 can be applied, and it completes the program that began with the observation that the futures price formulations Duffie (2001) and Karatzas and Shreve (1998) do not accommodate the Ho-Lee model. Obviously, analogous calculations apply model wherever one has lognormal bond prices and accumulation factors.

5 Recommendations and Further Examples

We suggest that for a rigorous but flexible theory of futures prices one should assume *both* condition (6) and condition (8). This double assumption accommodates every model that is covered by the formulations of Duffie (2001) and Karatzas and Shreve (1998), and, for parts of the theory of interest rate futures, this suggestion seems to provide one of the few viable alternatives.

Although the main example given here focused on a Gaussian models for the short rate, but there are many non-Gaussian models where the new framework also offers help. All term structure models which have positive interest rates automatically satisfy our condition (13) with $\alpha = 1$, and many such models have been introduce. Among these, the best known are probably the non-Gaussian models of Cox-Ingersoll-Ross type, such as those as described by Baxter and Rennie (1996, p. 157) or Björk and Landen (2002, p. 130), but one also has nonnegative interest rates in some more specialized models such as the modified proportional models of Heath, Jarrow, and Morton (1992, p. 95). In every case, the interest rates fail to be $\tilde{P} - a.s$ bounded, and, in several particular instances, the two-sided condition (4) also fails.

Finally, we should comment on the connection between Theorem 2 and the lognormal Black-Karanski model and the consol model examined in Hogan (1993) and Hogan and Weintraub (1993). For these models, the associated term structures have nonnegative interest rates, but Theorem 2 does not apply. Formally, this is because the terminal value need not be square integrable, but there are also *a priori* structural reasons. In particular, for the consol model Hogan (1993) finds that there can be zero prices for zero coupon bonds, so that model is not arbitrage free. In such a situation, one should not expect a martingale representation such as that of Theorem 2.

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