A REPEATED SIGNIFICANCE TEST FOR DISTRIBUTIONS WITH HEAVY TAILS

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Abstract. Repeated significance tests are frequently used in areas of science and technology in which the data is accumulated sequentially over time. In this article we develop a repeated significance test for independent and identically distributed observations from a continuous symmetric distribution with heavy tails, infinite variance and possibly no mean. This repeated significance test is nonparametric in nature as it is applicable to a class of distributions with a specified tail behaviour. We present an algorithm for selecting the continuation region associated with the repeated significance test that achieves specified significance level and power at a given alternative. Moreover, we derive approximations for the power function and expected sample size. Numerical results are presented to evaluate the performance of these approximations.

Key Words: Expected sample size; Invariance principle; Partial sums of truncated observations; Parabolic PDE; Power function for a repeated significance test; Significance level.

1. INTRODUCTION

Repeated significance tests were introduced in Armitage (1958) and since then have been discussed extensively in the statistical literature. Major developments in the theory and applications of repeated significance tests have been presented among others in: Dias and Garcia (1999), Hu (1988), Jennison and Turnbull (2000), Lai and Siegmund (1977 and 1979), Lalley (1983), Lerche (1986), Selke and Siegmund (1983), Sen (1981, 1985 and 1991), Siegmund (1982 and 1985), Takahashi (1990), Whitehead (1997), Woodroofe (1979 and 1982) and Woodroofe and Takahashi (1982). Repeated significance tests discussed in statistical literature are usually based on statistics having a normal or a t-distribution (Siegmund 1985 and Takahashi 1990). Nonparametric repeated significance tests have been discussed in Sen (1981, 1985 and 1991) for independent and identically distributed (iid) observations from a distribution in a class of distributions with a finite variance.

In this article a repeated significance test for iid observations from a continuous symmetric distribution with heavy tails and infinite variance and possibly no mean, as in the case of the Cauchy distribution, is devised. This repeated significance test is nonparametric in nature, and it is applicable to a class of distributions from a domain of attraction of a stable distribution. The method extends the approach in Sen (1981, 1985 and 1991) to situations when the average number of excluded terms is asymptotically negligible in comparison to the sample size.

To get convergence of the continuous time process associated with a sequence of partial sums of truncated observations to the standard Brownian motion, we need to delete only a small fraction of extreme observations. If we consider a random walk with Cauchy distributed increments and normalize it by its sample standard deviation, then the resulting random walk will not resemble a Brownian motion (Figure 1). This can be explained by the fact that the most significant input to the sequence of partial sums comes from a few extreme observations. If we exclude these extreme observations and construct the corresponding continuous time process based on partial sums of truncated observations, then this process will resemble a Brownian motion (Figure 2). Theorem 1 provides a precise theoretical statement that supports this observation. The closeness of the Brownian motion approximation to the continuous time process associated with the sequence of partial sums of truncated observations will allow us to construct a repeated significance test for a specified significance level (Section 2). The next two figures illustrate the main ideas used in evaluating an approximation for the power function for the repeated significance test discussed in this article. Figure 3 demonstrates the behavior of a Cauchy random walk with a constant drift. Since the behavior of partial sums of Cauchy random variables is influenced significantly by a few extreme observations, it is impossible to see the systematic bias that we have in this case (compare Figure 1 and Figure 3). However, if these extreme observations are excluded by resorting to partial sums of truncated random variables, we obtain a process that resembles a Brownian motion with a drift. The only difference from the classical situation is that the drift here is non linear. In Section 3 we will show how to evaluate that drift.

Distributions with heavy tails have been used in modeling computer network traffic (Crovella, Taqqu and Bestavros 1998 and Willinger, Paxson and Taqqu 1998), telecommunication systems (Crovella and Taqqu 1999), high frequency financial data (Müller, Dacorogna and Pictet 1998) and risk management and insurance data (Bassi, Embrecht and Kafetzaki 1998). In these areas of application the data is accumulated sequentially. Repeated significance tests can be used in the design and analysis of continuous monitoring schemes. In contrast to applications of sequential analysis in clinical trials (for example, Jennison and Turnbull (2000)) the sample sizes considered here are large. But this is the nature of the data sets mentioned above. In general, to infer that a distribution is heavy-tailed we should collect a substantial number of measurements.

The article is organized as follows. In Section 2, we develop a repeated significance test for the median of a continuous symmetric distribution with heavy tails in a class of stable distributions with a specified tail behavior. We show how to select a continuation region associated with this repeated significance test for specified significance level, power at a given alternative, initial sample size and target sample size. Our approach employs the Brownian motion approximation for partial sums of truncated random variables, based on the invariance principle in Pozdnyakov (2003). In Section 3, we develop an approximation for the power function of this repeated significance test by exploiting a link between the power function and a solution to a certain parabolic partial differential equation. The use of a two-dimensional degenerated diffusion and, as a consequence, parabolic partial differential equations (PDE) is a well-know theoretical approach to the problem of finding boundary crossing probabilities. However, to the best of our knowledge the technique presented in this article has not been used in the context of repeated significance tests. Moreover, in Section 3 a comparison with a RST based on a score statistic is presented. In Section 4, a numerical example is presented to illustrate the implementation of the repeated significance test based on the tables in De Long (1981). In Section 5, an approximation is developed for the expected sample size of the repeated significance test introduced in Section 2. A discussion of the results derived in this article is presented in Section 6.

2. REPEATED SIGNIFICANCE TEST: DESIGN AND IMPLEMENTATION

Let $\{X, X_i; i \geq 1\}$ be iid observations from a continuous distribution F symmetric about $-\infty < \theta < \infty$ and $\mathbf{E}(X^2) = \infty$. We are interested in developing a repeated significance test with initial sample size n_0 and target sample size N for testing

$$H_0 : \theta = 0 \text{ vs } H_a : \theta \neq 0. \tag{1}$$

In other words, we consider shift models $F(x - \theta), \theta \in \mathbb{R}$ and test $\theta = 0$. For the remaining of this section, unless stated otherwise, we assume that $H_0 : \theta = 0$ holds. Moreover, assume that F belongs to the domain of attraction of a stable distribution with exponent $0 < \gamma < 2$, i.e.

$$\mathbf{E}\left(X^2 \mathrm{I}_{(|X| \leq t)}\right) \sim t^{2-\gamma} L(t), \tag{2}$$

where L is a slowly varying function. Here and in what follows we denote by $u(t) \sim v(t)$ the asymptotic equivalence of two functions $u(t)$ and $v(t)$, in the sense that $\lim_{t\to\infty} [u(t)/v(t)] = 1$. We would like the repeated significance test to be based on a sequence of partial sums. The problem we encounter here is that the sequence of partial sums from a distribution with an infinite second moment does not converge to a Brownian motion. To overcome this difficulty we employ partial sums of truncated random variables. Let $\{d_n; n \geq 1\}$ be an increasing sequence of positive numbers such that

$$nP\left(|X| > d_n\right) \sim \gamma_n, \tag{3}$$

where $\gamma_n \nearrow \infty$, as $n \to \infty$. Define the truncated partial sums:

$$S_n = \sum_{i=1}^{n} X_i \mathrm{I}_{(|X_i| \leq d_n)}. \tag{4}$$

Denote by

$$B_n = Var(S_n). \tag{5}$$

The main theoretical difficulty we are facing here is that the sequence of truncated partial sums, $\{S_n; n \geq 1\}$, does not have independent increments and therefore the classical invariance principle (Donsker Theorem, Billingsley 1995, p. 520) is not applicable here. Let $\{W(t); 0 \leq t < \infty\}$ be the standard Brownian motion. The following invariance principle will be used to construct the continuation region for the proposed repeated significance test:

Theorem 1. *(Pozdnyakov 2003) If the random variable X belongs to the Feller class, i.e.*

$$\limsup_{t\to\infty} \frac{t^2 \mathbf{P}(|X| > t)}{\mathbf{E}\big(X^2 \mathrm{I}_{|X| \leq t}\big)} < \infty, \tag{6}$$

the average number of the excluded variables

$$n\mathbf{P}\big(|X| > d_n\big) \sim \gamma_n \nearrow \infty \tag{7}$$

and $\lim_{n\to\infty} B_n/B_{n+1} = 1$, *then* $S_n(t) \xrightarrow{d} W(t)$ *in the sense* $(\mathcal{C}[0,1], \rho)$, *where* $S_n(t)$ *is the linear interpolation between points*

$$\Big(0,0\Big), \Big(\frac{B_1}{B_n}, \frac{S_1}{\sqrt{B_n}}\Big), ..., \Big(1, \frac{S_n}{\sqrt{B_n}}\Big).$$

Since B_n is usually unknown, following Sen (1981, p. 249), we replace it with an almost sure equivalent sequence of random variables. For the problem at hand we choose:

$$A_n = \sum_{i=1}^{n} X_i^2 \mathrm{I}_{(|X_i| \le d_n)} - \frac{S_n^2}{\sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}}. \tag{8}$$

The following result supports the use of this sample version of B_n.

Proposition 1. *Assume that the conditions of Theorem 1 hold, and additionally:*

$$\lim_{n\to\infty} \frac{n\mathbf{P}\big(|X| > d_n\big)}{\ln\ln(n)} = \infty \tag{9}$$

and

$$\ln\ln(B_n) = o(n). \tag{10}$$

Then

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1 \ a.s.. \tag{11}$$

Proof. From Egorov and Pozdnyakov (1997) we know that under the conditions of the proposition the strong law of large numbers for the truncated sums of squares holds:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i^2 \mathrm{I}_{(|X_i| \le d_n)}}{B_n} = 1 \text{ a.s..} \tag{12}$$

Therefore, to prove (11) we need to show that

$$\frac{S_n^2}{B_n \sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}} \to 0 \text{ a.s..}$$

Egorov and Pozdnyakov (1997) also proved the iterated logarithm law for the truncated sums S_n:

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2B_n \ln\ln(B_n)}} = 1 \text{ a.s..} \tag{13}$$

Let m be such that $\mathbf{P}(|X| \le d_m) > 0$. By the classical strong law of large numbers we get that

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}}{n} \ge \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_m)}}{n} = \mathbf{P}(|X| \le d_m),$$

i.e.

$$\frac{n}{\sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}} = O(1).$$

Hence, by using (10) we find that

$$\frac{S_n^2}{B_n \sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}} = \frac{S_n^2}{B_n \ln\ln(B_n)} \times \frac{B_n \ln\ln(B_n)}{B_n \sum_{i=1}^{n} \mathrm{I}_{(|X_i| \le d_n)}} = o(1) \text{ a.s..}$$

□

In this article we investigate the performance of the repeated significance test introduced below. Let

$$\tau = \min\left\{n_0 \le n \le N; |S_n| \ge b\sqrt{A_n}\right\} \tag{14}$$

be the stopping time associated with the repeated significance test, where n_0 is the initial sample size and N is the target sample size. The repeated significance test stops and rejects H_0, given in equation (1), if and only if $\tau \le N$. For all values of θ, the power function for the repeated significance test is given by:

$$\pi(\theta) = \mathbf{P}_\theta\left(\tau \le N\right) = 1 - \beta(\theta), \tag{15}$$

where

$$\beta(\theta) = \mathbf{P}_\theta\left(|S_n| < b\sqrt{A_n}; n_0 \le n \le N\right), \tag{16}$$

is the probability of type II error function and $\left\{b_n = b\sqrt{A_n}; n_0 \le n \le N\right\}$ is a sequence of constants that determine the continuation region of the repeated significance test. The significance level of this test is given by:

$$\begin{aligned}\alpha &= \pi(0) = \mathbf{P}_0\left(\tau \le N\right) = 1 - \beta(0)\\ &= \mathbf{P}_0\left\{\max_{n_0 \le n \le N}\left(\frac{|S_n|}{\sqrt{A_n}}\right) \ge b\right\}.\end{aligned} \tag{17}$$

To implement this test we need an accurate approximation for $b = b(\alpha, n_0, N)$ (the critical value of the test statistic associated with the repeated significance test), for a specified value of $0 < \alpha < 1$. The test statistic and the continuation region associated with this repeated significance test depend on the truncating levels d_n and the ratio of the initial and target sample sizes, which can be determined via the following result.

Proposition 2. *Assume that F belongs to the domain of attraction of a continuous symmetric stable distribution with exponent $0 < \gamma < 2$. Then the following results are true:*
1) F belongs to the Feller class.
2) The average number of the excluded terms $n\mathbf{P}(|X| > dn^\delta) \nearrow \infty$ whenever $1 - \gamma\delta > 0$. In particular, any $0 < \delta < 1/2$ guarantees it for all $0 < \gamma < 2$.
3) If $1 - \gamma\delta > 0$ and $\lim_{n_0, N \to \infty}(n_0/N) = c < 1$, then

$$\max_{n_0 \le n \le N} \frac{|S_n|}{\sqrt{A_n}} \xrightarrow{d} \sup_{[t_0 \le t \le 1]} \frac{|W(t)|}{\sqrt{t}},$$

where

$$t_0 = c^{1+(2-\gamma)\delta}. \tag{18}$$

Proof. It follows from Feller (1971, p. 313) that (2) is equivalent to

$$\lim_{t\to\infty} \frac{t^2\mathbf{P}(|X| > t)}{\mathbf{E}\left(X^2 \mathrm{I}_{(|X| \le t)}\right)} = \frac{2-\gamma}{\gamma}. \tag{19}$$

Thus, the random variable X belongs to the Feller class.

When $0 < \gamma < 2$ we have that

$$\mathbf{P}(|X| > t) \sim \frac{2-\gamma}{\gamma} t^{-\gamma} L(t).$$

Hence, by (19) we get

$$n\mathbf{P}(|X| > dn^{\delta}) \sim \frac{2-\gamma}{\gamma} d^{-\gamma} n^{1-\gamma\delta} L(dn^{\delta}).$$

Recall that if L is a slowly varying function then

$$x^{-\epsilon} < L(x) < x^{\epsilon}$$

for any fixed $\epsilon > 0$ and all x sufficiently large. Therefore, the average number of the excluded terms $n\mathbf{P}(|X| > dn^{\delta})$ goes to the infinity if $1 - \gamma\delta > 0$.

Suppose that $n_0, N \to \infty$ and $n_0/N \to c < 1$. It follows from (2) that

$$B_n \sim nd^{2-\gamma} n^{(2-\gamma)\delta} L(dn^{\delta}).$$

By the definition of a slowly varying function we get

$$\lim_{n_0, N \to \infty, \frac{n_0}{N} \to c} \frac{L(dn_0^{\delta})}{L(dN^{\delta})} = 1,$$

which implies that

$$\lim_{n_0, N \to \infty, \frac{n_0}{N} \to c} \frac{B_{n_0}}{B_N} = c^{1+(2-\gamma)\delta}.$$

Therefore, if $d_n = dn^{\delta}$, where $1 - \gamma\delta > 0$, $n_0, N \to \infty$ and $n_0/N \to c < 1$ then by Theorem 1 we can conclude that

$$\max_{n_0 \le n \le N} \frac{|S_n|}{\sqrt{A_n}} \xrightarrow{d} \sup_{[c^{1+(2-\gamma)\delta}, 1]} \frac{|W(t)|}{\sqrt{t}}.$$

□

In view of Proposition 2, set the truncating levels $d_n = dn^{\delta}$, where $d > 0$ and $0 < \delta < 1/2$. Let $c = n_0/N$, be the ratio of the initial and target sample sizes of the repeated significance test. It follows from Proposition 2 that $b = b(\alpha, n_0, N)$ can be approximated by $b_{t_0}(\alpha)$ by solving

$$\mathbf{P}\left(\sup_{[t_0, 1]} \frac{|W(t)|}{\sqrt{t}} > b_{t_0}(\alpha) \right) = \alpha,$$

where t_0 is given in equation (18). We use De Long (1981) to evaluate $b_{t_0}(\alpha)$.

Example 1. *Domain of attraction of a Cauchy distribution with location parameter θ and scale parameter* 1.

Let $\{X_i; i \ge 1\}$ be a sequence of iid observations from a distribution F in the domain of attraction of a Cauchy distribution with location parameter θ and scale parameter 1. Assume that $H_0 : \theta = 0$ is true. We consider here truncation levels $d_n = n^{1/4}, n_0 \le n \le N$. In Table 1, the performance of the proposed repeated significance test is evaluated in terms of accuracy of achieving an assigned significance level α, for given values of n_0 and N. The theoretical critical values $b_{t_0}(\alpha)$ and the corresponding targeted significance levels have been obtained from De Long (1981). The achieved significance levels were evaluated from a simulation with 10,000 trials.

Note that the achieved significance levels are close to their targeted values even for a moderate value of $n_0 = 30$, as long as the significance level is not too small. A value of $n_0 = 100$ produced quite accurate results even for $\alpha = .01$. The relative error for $n_0 = 100$ is less than 10%. In the case of $n_0 = 30, \alpha = .1$ the error is

Table 1. Simulated Significance Levels

n_0	N	t_0	$b_{t_0}(\alpha)$	*targeted* α	*simulated* α
100	303	1/4	2.7	.0503	.0541
100	303	1/4	3.3	.0098	.0094
100	754	1/12.5	2.6	.0989	.1012
30	91	1/4	2.7	.0503	.0638
30	91	1/4	3.3	.0098	.0167
30	226	1/12.5	2.6	.0989	.1119

about 10%. For significance levels less than .05 we need to employ a higher initial value to get accurate approximations.

3. POWER CALCULATIONS

In this section we construct an approximation for the power function of the repeated significance test. First we show that the power function can be approximated by the probability of exit of a degenerated diffusion process from a certain domain. Then we demonstrate how the exit probability can be found via exploiting a link between the degenerated diffusion process and a parabolic partial differential equation.

3.1. **Brownian Motion with a Drift Approximation.** Let $\{X, X_i; i \geq 1\}$ be iid observations from an absolutely continuous distribution F symmetric about $-\infty < \theta < \infty$ and $\mathbf{E}(X^2) = \infty$. Assume that

$$F'(t) \sim (2-\gamma)Kt^{-1-\gamma}, \text{ as } t \to \infty, \tag{20}$$

where $0 < \gamma < 2$ and K are known. As a consequence, we also have

$$\mathbf{E}\left(X^2 \mathrm{I}_{(|X|<t)}\right) \sim Kt^{2-\gamma}. \tag{21}$$

The truncation level is chosen to be $d_n = dn^\delta$. The truncation under the alternative $H_a : \theta \neq 0$, leads to the sum S_n, which for large n will be approximated by

$$S_n^0 + \theta n, \tag{22}$$

where

$$S_n^0 = \sum_{i=1}^{n} (X_i - \theta)\mathrm{I}_{(|X_i - \theta| \leq d_n)}.$$

Since,

$$S_n^0 = \sum_{i=1}^{n} (X_i - \theta)\mathrm{I}_{(|X_i - \theta| \leq d_n)} = \sum_{i=1}^{n} X_i \mathrm{I}_{(|X_i - \theta| \leq d_n)} - \theta \sum_{i=1}^{n} \mathrm{I}_{(|X_i - \theta| \leq d_n)},$$

we get that

$$\begin{aligned} S_n =& S_n^0 + S_n - S_n^0 \\ =& S_n^0 - \Delta_n + \theta \sum_{i=1}^{n} \mathrm{I}_{(|X_i - \theta| \leq d_n)}, \end{aligned}$$

where $\Delta_n = \sum_{i=1}^n X_i \mathrm{I}_{(|X_i - \theta| \leq d_n)} - \sum_{i=1}^n X_i \mathrm{I}_{(|X_i| \leq d_n)}$.

The following proposition supports approximating S_n by $S_n^0 + \theta n$.

Proposition 3. *Let X be a random variable with an absolutely continuous distribution symmetric about θ that satisfies (20). If $\delta > 1/(2+\gamma)$ then*

$$\mathbf{E}\left(\frac{|\Delta_n - \theta\sum_{i=1}^n \mathrm{I}_{(|X_i-\theta|\le d_n)} + \theta n|}{\sqrt{B_n}}\right) \to 0.$$

Proof. First let us show that

$$\mathbf{E}\left(\frac{n - \sum_{i=1}^n \mathrm{I}_{(|X_i-\theta|\le d_n)}}{\sqrt{B_n}}\right) = \mathbf{E}\left(\frac{\sum_{i=1}^n \mathrm{I}_{(|X_i-\theta|> d_n)}}{\sqrt{B_n}}\right) \to 0$$

Indeed,

$$\begin{aligned}
\mathbf{E}\left(\frac{\sum_{i=1}^n \mathrm{I}_{(|X_i-\theta|> d_n)}}{\sqrt{B_n}}\right) &= \frac{n\mathbf{P}(|X-\theta|>d_n)}{\sqrt{B_n}} \\
&= O\left(\frac{nd_n^{-\gamma}}{n^{1/2}d_n^{1-\gamma/2}}\right) \\
&= O\left(\frac{n^{1/2}}{n^{\delta(1+\gamma/2)}}\right) \to 0
\end{aligned}$$

Now let us show that $\mathbf{E}(\Delta_n/\sqrt{B_n}) \to 0$. Without loss of generality assume that $\theta > 0$. In this case $\Delta_n \ge 0$ and

$$\begin{aligned}
\Delta_n &= \sum_{i=1}^n X_i \mathrm{I}_{(|X_i-\theta|\le d_n)} - \sum_{i=1}^n X_i \mathrm{I}_{(|X_i|\le d_n)} \\
&= \sum_{i=1}^n X_i \mathrm{I}_{(X_i\in(d_n, d_n+\theta])} - X_i \mathrm{I}_{(X_i\in[-d_n,-d_n+\theta))} \\
&\le (d_n+\theta)\sum_{i=1}^n \mathrm{I}_{(X_i\in[-d_n,-d_n+\theta)\cup(d_n,d_n+\theta])}.
\end{aligned}$$

Since the random variable X is symmetric around θ we have that

$$\mathbf{P}\big(X_i \in [-d_n, -d_n+\theta) \cup (d_n, d_n+\theta]\big) = \mathbf{P}\big(X \in (d_n, d_n+2\theta]\big).$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left(\frac{\Delta_n}{\sqrt{B_n}}\right) &\le \frac{d_n+\theta)n\mathbf{P}\big((d_n, d_n+2\theta]}{\sqrt{B_n}} \\
&= O\left(\frac{nd_n d_n^{-1-\gamma}}{d_n^{1-\gamma/2}}\right) \\
&= O\left(\frac{n^{1/2}}{n^{\delta(1+\gamma/2)}}\right) \to 0.
\end{aligned}$$

□

In view of this approximation for S_n, and the fact that as $n \to \infty$

$$\frac{B_n}{B_N} \sim \frac{nKd^{2-\gamma}n^{\delta(2-\gamma)}}{NKd^{2-\gamma}N^{\delta(2-\gamma)}} = \left(\frac{n}{N}\right)^{1+\delta(2-\gamma)},$$

we obtain:

$$\frac{S_n}{\sqrt{B_N}} \approx \frac{S_n^0 + \theta n}{\sqrt{B_N}} \approx \frac{S_n^0}{\sqrt{B_N}} + \theta \frac{n}{K^{1/2} d^{(2-\gamma)/2} N^{(1+\delta(2-\gamma))/2}}.$$

The sign $\approx$ is used here to indicate a practical approximation motivated by Proposition 3 and supported by numerical examples presented below.

Let $t = (n/N)^{1+\delta(2-\gamma)}$, then the drift term can be written as

$$\theta \frac{n}{K^{1/2} d^{(2-\gamma)/2} N^{(1+\delta(2-\gamma))/2}} = \theta \frac{N^{(1-\delta(2-\gamma))/2}}{K^{1/2} d^{(2-\gamma)/2}} t^{1/(1+\delta(2-\gamma))}.$$

Therefore, under H_a and for large n, the continuous process associated with S_n is a Brownian motion with a nonlinear drift:

$$W(t) + \theta \frac{N^{(1-\delta(2-\gamma))/2}}{K^{1/2} d^{(2-\gamma)/2}} t^{1/(1+\delta(2-\gamma))}.$$

The power function given by (15) and (16) can be approximated by:

$$\pi(\theta) \approx 1 - \mathbf{P}\Big(\Big| W(t) + \theta \frac{N^{(1-\delta(2-\gamma))/2}}{K^{1/2} d^{(2-\gamma)/2}} t^{1/(1+\delta(2-\gamma))} \Big| < b_{t_0}(\alpha)\sqrt{t} \text{ for all } t \in [t_0, 1] \Big). \tag{23}$$

3.2. **Link to a PDE.** It follows from (23), that approximating the power function of the repeated significance test amounts to evaluating:

$$\mathbf{P}\left(|W(t) + \kappa t^\rho| < b\sqrt{t} \text{ for all } t \in [t_0, 1] \right),$$

where

$$b = b_{t_0}(\alpha), \quad \kappa = \theta \frac{N^{(1-\delta(2-\gamma))/2}}{K^{1/2} d^{(2-\gamma)/2}}, \text{ and } 1/2 < \rho = 1/(1 + \delta(2-\gamma)) < 1.$$

Since,

$$\mathbf{P}\left(-b\sqrt{t} - \kappa t^\rho < W(t) < b\sqrt{t} - \kappa t^\rho \text{ for all } t \in [t_0, 1] \right) = \int_{-b\sqrt{t_0} - \kappa t_0^\rho}^{b\sqrt{t_0} - \kappa t_0^\rho} \psi(x) \mathbf{P}\big(W(t_0) \in dx\big),$$

where

$$\psi(x) = \mathbf{P}\left(-b\sqrt{t} - \kappa t^\rho < W(t) < b\sqrt{t} - \kappa t^\rho \text{ for all } t \in [t_0, 1] \Big| W(t_0) = x \right).$$

Approximating the power function is equivalent to solving the following problem.

Consider the domain D

$$\big\{(x, y) : -t_0 \le y \le 1 - t_0, \quad -b\sqrt{t_0 + y} - \kappa(t_0 + y)^\rho < x < b\sqrt{t_0 + y} - \kappa(t_0 + y)^\rho \big\}.$$

Let $\tau_D(x, y)$ be a first time when the degenerated two-dimensional diffusion $(x_t, y_t) = (x + W(t), y + t)$ exits from the domain D, where (x, y) belongs to the interior of the domain D. What is the probability that a Brownian motion starting at point x and at time y will stay inside the curved boundaries, i.e.

$$\mathbf{P}\big(y_{\tau_D(x,y)} = 1 - t_0\big)?$$

The generating operator of the diffusion (x_t, y_t) is given by

$$\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}.$$

By Venttsel (1996, p. 333) the function

$$v(x,y) = \mathbf{P}\big(y_{\tau_D(x,y)} = 1 - t_0\big)$$

is the unique solution of the PDE

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(x,y) + \frac{\partial v}{\partial y}(x,y) = 0 \quad (x,y) \in D,$$

that satisfies the following boundary conditions:
1. $v(\pm b(t_0+y)^{1/2} - \kappa(t_0+y)^\rho, y) \equiv 0$,
2. $v(x, 1-t_0) \equiv 1$.

By solving this parabolic partial differential equation numerically we can evaluate an approximation for the power function.

3.3. **Numerical Computations.** To solve the PDE we used the nonlinear solver `pdenonlin` (`full` Jacobian based evaluation) which is a standard function of the MATLAB PDE Toolbox. First we tested this method in the situation when the drift parameter θ is equal to 0. In this case we have a well-known problem of crossing a square-root boundary by the standard Brownian Motion. To get probabilities that deviate from the corresponding probabilities presented by De Long (1981) by less than .001 we need to refine the initial mesh four times.

Table 2. Approximations for the Power Function, $N = 400$

θ	BM approximation (4 refinements)	Simulation (1,000 trials)
0	.0508 (.0503)	.049
.25	.1611	.137
.5	.4906	.482
.75	.8320	.839
1	.9750	.964

In Table 2 we present approximations for the power function for various choices of θ computed via the Brownian motion approximation and by simulations. Again we consider here the class of distributions in the domain of attraction of a Cauchy distribution with location parameter θ and scale parameter 1. The initial sample size n_0 is 159, the target sample size N is 400, and the critical value $b_{t_0}(\alpha) = 2.7$ ($\alpha \approx .05$). This choice corresponds to the first row of Table 1. However, in this case we choose a higher truncating level $d_n = dn^\delta = 2 \times n^{1/2}$. The multiplier $d = 2$ is taken in order to get a good approximation in formula (21) (recall that for the Cauchy distribution $K = 2/\pi$ and $\gamma = 1$). Note, that the multiplier d, asymptotically, does not effect the type I error (see Proposition 2).

The difference between simulated values of the power function and its approximation via BM is satisfactory, especially if we take into account that the error in formula (21) is about 10%. The value .0503 in parenthesis is evaluated using De Long (1981).

Table 3 presents similar computational results. Here n_0 is 397, and N is 1000, i.e. the target sample size is larger, but the ratio n_0/N is kept the same, so t_0 does not change.

TABLE 3. Approximations for the Power Function, $N = 1000$

θ	BM approximation (4 refinements)	Simulation (1,000 trials)
0	.0508	.044
.125	.0930	.081
.25	.2282	.227
.375	.4453	.427
.5	.6848	.680

3.4. **Comparison to the RST Based on Score Statistic.** It is interesting to see how much power is lost by the RST based on truncated sums with respect to some other (parametric) tests. For instance, if it is known that the distribution of X is Cauchy with scale parameter 1, then one can use the score statistic (see, for example, Lehmann (1999, p. 529)):

$$T_n = \sum_{i=1}^{n} \frac{2X_i}{1 + X_i^2}.$$

Since the random variable $2X_i/(1 + X_i^2)$ is a bounded random variable the classical Donsker Theorem holds for T_n. Table 4 shows that the score RST achieves the power 1 at $\theta = .25$, while the truncated sum RST has power close to 1 at $\theta = .75$.

TABLE 4. Comparison, Cauchy Distribution, 1000 trials

θ	Truncated Sum RST $n_0 = 397, N = 1000, t_0 = 1/4$ $b_{t_0}(\alpha) = 2.7, \alpha \approx .05$	Score RST $n_0 = 250, N = 1000, t_0 = 1/4$ $b_{t_0}(\alpha) = 2.7, \alpha \approx .05$	
	$d_n = n^{1/2}$	Normalized by $n/2$	Self-normalized
0	.045	.040	.048
.125	.119	.663	.681
.25	.395	1	.998
.5	.916	1	1
.75	1	1	1

TABLE 5. Comparison to the Score RST, Cauchy+Uniform[-5,5], $N = 1000$, 1000 trials

θ	Truncated Sum RST $n_0 = 397, N = 1000, t_0 = 1/4$ $b_{t_0}(\alpha) = 2.7, \alpha \approx .05$	Score RST $n_0 = 250, N = 1000, t_0 = 1/4$ $b_{t_0}(\alpha) = 2.7, \alpha \approx .05$	
	$d_n = n^{1/2}$	Normalized by $n/2$	Self-normalized
0	.047	.023	.055
.25	.238	.052	.104
.5	.779	.187	.295
.75	.982	.474	.639

The loss of power is a natural trade-off for the non-parametric approach that we use. To get the right α level for the truncated sum RST we need to know only the tail behaviour of the distribution. The score RST takes into account the entire distribution. If the distribution is not exactly Cauchy, the performance of the score RST can deteriorate. For instance, if X has a distribution which is the convolution of the Cauchy and the Uniform on $[-5, 5]$, the tail behaviour is asymptotically equivalent to that of Cauchy. Table 6 shows that the RST presented in this article is more powerful than the score RST. Of course, in the case when we are not sure whether the distribution under consideration is exactly Cauchy a self-normalized score statistics should be used.

4. A NUMERICAL EXAMPLE

In this section we describe an algorithm for designing and implementing a repeated significance test with specified significance level and power at a specified alternative. For simplicity, we illustrate the algorithm for an example with data from a Cauchy distribution, i.e. $K = 2/\pi$ and $\gamma = 1$. Suppose that

1. the monitoring starts at $n_0 = 150$,
2. the targeted significance level is about .05,
3. the targeted power at $\theta = .5$ is about .90,
4. the truncation level $d_n = n^{1/2}$.

The first step of the algorithm obtains from the tables in De Long (1981) plausible values of t_0 and $b_{t_0}(.05)$. The second step uses Proposition 2 and the initial sample size $n_0 = 150$, to obtain the appropriate target sample size N. In the final step we employ the method developed in Section 3 to evaluate the power of the repeated significance test at $\theta = .5$, for selected values of t_0 and $b_{t_0}(.05)$. The design parameters that satisfy the targeted requirements are chosen for the implementation of the testing procedure. The numerical results associated with the algorithm described above are summarized in Table 6.

TABLE 6. Test Design

t_0	α	$b_{t_0}(\alpha)$	n_0	N	$\pi(.5)$
1/2.5	.0491	2.6	150	276	.7096
1/4	.0503	2.7	150	377	.7736
1/7.5	.0513	2.8	150	574	.8509
1/15	.0499	2.9	150	912	.9161

Based on the numerical results presented in Table 6 we will choose a repeated significance test with $n_0 = 150$, $N = 912$ and $b = 2.9$. A simulation with 1000 trials from a centered Cauchy distribution using these design parameters indicated an achieved significance level of .042. A simulation with 1000 trials from a Cauchy distribution with median $\theta = .5$ for these design parameters resulted with an achieved power of .883.

5. EXPECTED SAMPLE SIZE

The expected sample size is an important characteristic of a sequential procedure. In this section we develop an approximation for the expected stopping time, $\mathbf{E}(\tau)$, associate with the repeated significance test developed in Section 2, under the assumptions on the distribution given in Section 3.

5.1. **Another PDE.** We show here how a similar PDE technique can be employed to approximate the expected sample size for a given parameter θ. We start from the corresponding Brownian motion with a nonlinear drift that is associated with our testing procedure. First, we find the expected stopping time for the continuous counterpart. Next, we get an approximation of the expected sample size, and then finally check this approximation numerically.

Let the domain D and stopping time $\tau_D(x, y)$ be the same ones used in Section 3. Then the stopping time τ^* for the continuous repeated significance test is given by

$$\mathbf{E}(\tau^*) = t_0 + \int_{-b\sqrt{t_0}-\kappa t_0^\rho}^{b\sqrt{t_0}-\kappa t_0^\rho} \mathbf{E}(\tau_D(x, 0))\mathbf{P}\big(W(t_0) \in dx\big). \tag{24}$$

Hence, to evaluate $\mathbf{E}(\tau^*)$ we need to compute $\mathbf{E}(\tau_D(x, 0))$. According to Venttsel (1996, p. 333), the function

$$v(x, y) = \mathbf{E}(\tau_D(x, y))$$

is the unique solution of the following PDE:

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial v}{\partial y}(x, y) = -1 \quad (x, y) \in D,$$

that satisfies the following boundary conditions:
1. $v(\pm b(t_0 + y)^{1/2} - \kappa(t_0 + y)^\rho, y) \equiv 0$,
2. $v(x, 1 - t_0) \equiv 0$.

If we compare this PDE to the one in Section 3, we note two distinctions: the right side is now -1 (was 0) and we have 0 boundary conditions everywhere. This PDE can be easily solved numerically by standard MATLAB tools.

5.2. **Expected Sample Size Approximation.** Note that in the case of the functional central limit theorem the natural time scale is given in terms of the variance of truncated sums $B_n \sim nKd^{2-\gamma}n^{\delta(2-\gamma)}$, therefore, we have the following approximate relationship between τ and τ^*:

$$\tau^* \sim \frac{B_\tau}{B_N} \sim \left(\frac{\tau}{N}\right)^{1+\delta(2-\gamma)}. \tag{25}$$

We can see that we have a nonlinear relation between τ^* and τ, so the expected value $\mathbf{E}(\tau)$ cannot be expressed in terms of $\mathbf{E}(\tau^*)$. Still, one can employ the following first order Taylor approximation,

$$\mathbf{E}(\tau) \sim \mathbf{E}\big[\tau^{*1/(1+\delta(2-\gamma))}N\big] \approx \big[\mathbf{E}(\tau^*)\big]^{1/(1+\delta(2-\gamma))}N.$$

5.3. **Numerical results.** Here we consider again the Cauchy distribution. The initial sample size $n_0 = 159$, the target sample size $N = 400$, the truncation level $d_n = 2n^{1/2}$, and the critical value $b_{t_0}(\alpha) = 2.7$ ($\alpha \approx .05$).

Numerical results in Table 7 indicate that the Brownian motion approximation provides an useful approximation for the expected sample size associated with the repeated significance test.

Table 7. Expected Sample Size

θ	BM approximation (4 refinements)	Simulation (1000 trials)
.25	377	375
.5	321	318
.75	241	236
1	184	182

6. DISCUSSION

In this article we have derived a repeated significance test for observations from a symmetric continuous heavy tailed distribution with an infinite variance. The implementation of this testing procedure is based on an invariance principle for partial sums of truncated observations. This enabled us to design a repeated significance test that is targeted to achieve the designated significance level. Numerical results presented in this accurate support this method. We also developed a new algorithm to approximate the power function and the expected stopping time associated with the repeated significance test. It is based on solutions of certain parabolic stochastic partial differential and their link to the power function and the expected stopping time of the repeated significance test. Numerical results presented in this article show that these approximations are quite accurate.

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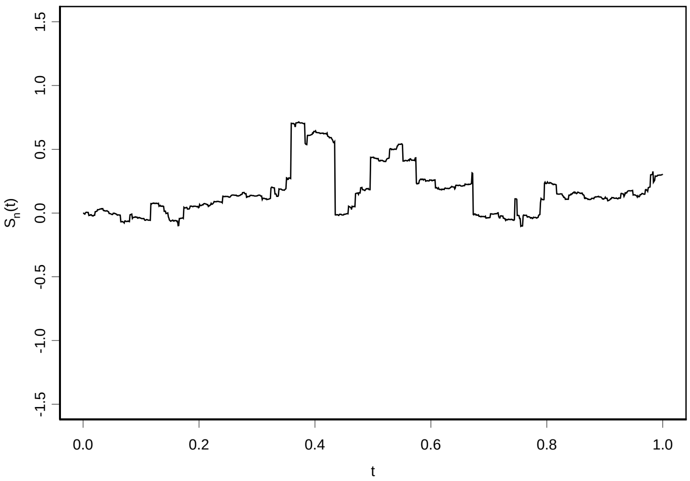

FIGURE 1. Cauchy random walk normalized by its sample standard deviation, $n = 1000$

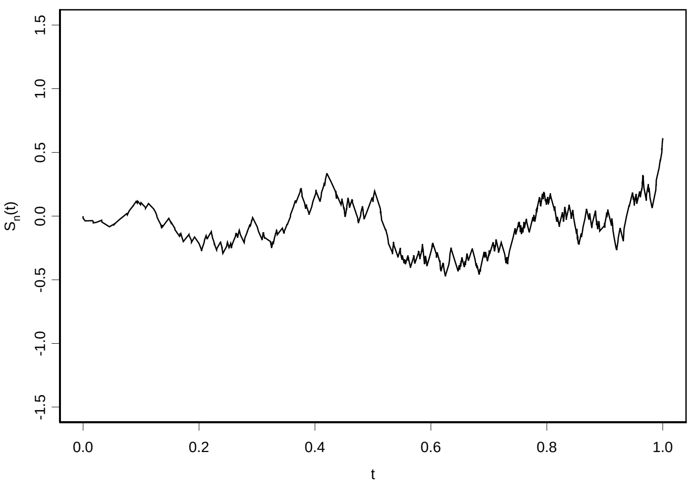

FIGURE 2. Truncated Cauchy random walk normalized by its sample standard deviation, $n = 1000$

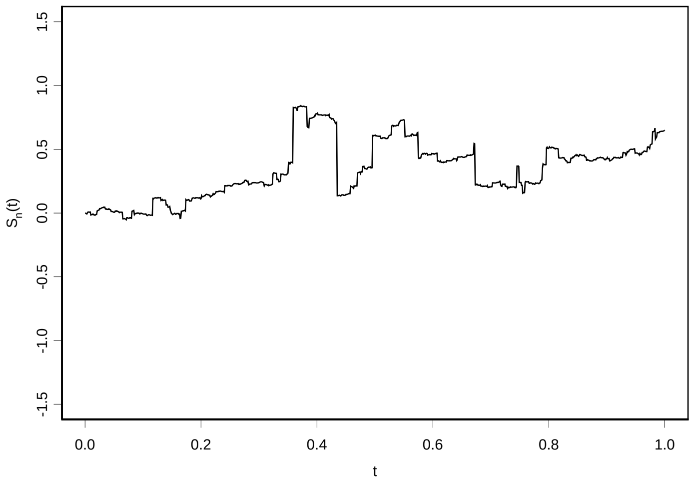

FIGURE 3. Noncentered ($\theta = .2$) Cauchy random walk normalized by its sample standard deviation, $n = 1000$

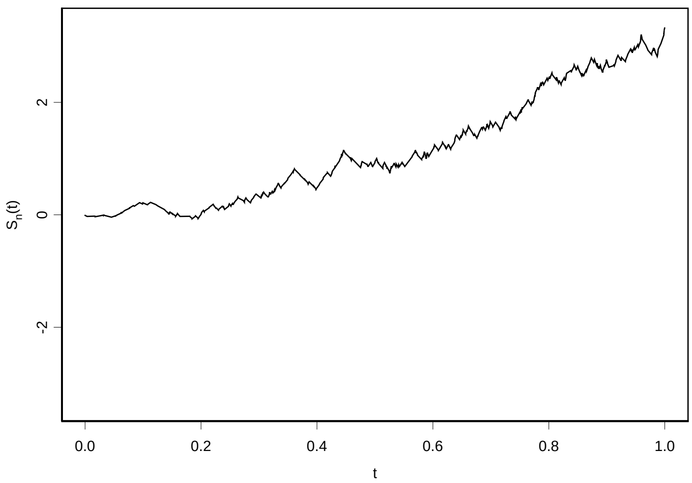

FIGURE 4. Truncated noncentered ($\theta = .2$) Cauchy random walk normalized by its sample standard deviation, $n = 1000$