CONVEXITY BIAS IN THE PRICING OF EURODOLLAR SWAPS

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ABSTRACT. The traditional use of LIBOR futures prices to obtain surrogates for the Eurodollar forward rates is proved to yield a systematic bias in the pricing of Eurodollar swaps when one assumes that the yield curve is well described by the Heath-Jarrow-Morton model. The resulting theoretical inequality is consistent with the empirical observations of Burghardt and Hoskins (1995), and it provide a theoretical basis for price anomalies that are suggested by more recent empirical data.

1. INTRODUCTION

The prices of Eurodollar swaps are uniquely determined by the value of the Eurodollar forward rates, and the main problem in pricing a Eurodollar swap comes from the unfortunate fact that the Eurodollar forward rates are not directly observable. This has led to the uneasy custom among market practitioners to use LIBOR futures prices in order to calculate surrogates for the missing forward rates. It has long been understood that the daily settlement of futures contracts implies that these surrogates are imperfect, yet much remains to be discovered about the true nature of the biases that may be introduced when these surrogates are used to price Eurodollar swaps.

The main goal here is to show that under a reasonably general model for the term structure of interest rates, one can prove that there is a systematic bias in the customary process for pricing interest rate swaps. Our result provides a theoretical confirmation of the empirical observations of Burghardt and Hoskins (1995), and it also provides a basis for the more precise analysis of swap prices.

The principal results are developed in Section 4, but, before those results can be derived, we need to introduce some notation and to recall some familiar properties of the Heath-Jarrow-Morton term structure model. We then develop a technical result that implies the HJM framework typically generates a term structure model where futures rates are systematically higher than the corresponding forward rates. Section 4 then uses this technical result to obtain results on the biases that are produced from using futures rates as surrogates for forward rates in the pricing of zero-coupon bonds and swaps.

Finally, Section 5 provides a brief analysis of the recent empirical behavior of swap rates and their relationship to the *a priori* bounds obtained here. The data suggest several engaging anomalies, and, in particular, one finds that there is at

1991 *Mathematics Subject Classification.* Primary: 91B28; Secondary: 60H05, 60G44.

Key words and phrases. Heath-Jarrow-Morton model, HJM model, interest rates, LIBOR, futures prices, arbitrage pricing, swap, equivalent martingale measures.

least modest evidence that the arbitrage opportunities suggested by Burghardt and Hoskins may still survive.

2. Background on the HJM Model

Our analysis is based on a term structure model of Heath, Jarrow, and Morton (1992) that has become one of the standard tools for the theoretical analysis of fixed income securities and their associated derivatives. This model is well discussed in several recent books, such as Baxter and Rennie (1996), Duffie (1996), and Musiela and Rutkowski (1997)), but some review of the HJM model seems useful here in order to set notation and to keep our derivation of the pricing inequality reasonably self-contained.

If $P(t,T)$ denotes the price at time t of a bond that pays one dollar at the maturity date $T \le \tau$, then the first step in the construction of an HJM model is the assumption that $P(t,T)$ has an integral representation,

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right) \qquad 0 \le t \le T \le \tau, \tag{1}$$

where the processes $\{f(t,T) : 0 \le t \le T \le \tau\}$ model the *instantaneous forward rate* that should reflect the interest rate available at time t for a riskless loan that begins at date T and which is paid back "an instant" later. Moreover, under the HJM model, one further assumes that $f(t,T)$ may be written as a stochastic integral

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \boldsymbol{\sigma}(u,T)^{\perp} d\boldsymbol{B}_u, \tag{2}$$

where $\boldsymbol{B}_t$ denotes an n-dimensional Brownian motion and the two processes

$$\{\alpha(u,T) : 0 \le u \le T \le \tau\} \quad \text{and} \quad \{\boldsymbol{\sigma}(u,T) : 0 \le u \le T \le \tau\}$$

are respectively $\mathbb{R}$ and $\mathbb{R}^n$ valued and adapted to the standard filtration $\mathcal{F}_t$ of $\{\boldsymbol{B}_t\}$. Also, one should note that the symbol $\perp$ in the second integral of (2) denotes the vector transpose, and both of the processes $\boldsymbol{B}_t$ and $\boldsymbol{\sigma}(u,T)$ are viewed as column vectors.

The representation (1) imposes almost no real constraint on $P(t,T)$ except non-negativity and the normalization $P(T,T) = 1$. In fact, the essential nature of the HJM model only becomes evident once one restricts attention to a subclass of instantaneous forward rates $f(t,T)$ for which one can guarantee that there are no arbitrage opportunities between bonds of differing maturities.

The Forward Rate Drift Restriction

An essential feature of the HJM model is that in almost any economically meaningful context, the coefficient processes $\alpha(t,T)$ and $\boldsymbol{\sigma}(t,T)$ of the SDE for $f(t,T)$ must satisfy a certain simple analytical relationship. Specifically, one knows that $\alpha(t,T)$ may be assumed to be of the form

$$\alpha(t,T) = \boldsymbol{\sigma}(t,T)^{\perp}\big[\gamma(t) + \int_t^T \boldsymbol{\sigma}(t,u)\,du\big] \tag{3}$$

where $\boldsymbol{\gamma}(t)$ is an adapted n-dimensional process such that

$$E \exp\left(-\int_0^\tau \boldsymbol{\gamma}(u)^\perp d\boldsymbol{B}_u - \frac{1}{2}\int_0^\tau |\boldsymbol{\gamma}(u)|^2\, du\right) = 1. \tag{4}$$

When this identity holds, we say that $f(t,T)$ satisfies the *forward rate drift restriction*, and we call the process $\boldsymbol{\gamma}(t)$ that appears in this formula the *market price for risk*. The economic motivation from this condition comes from the fact that it implies several useful properties of the probability measure $\widetilde{P}$ defined on $\mathcal{F}_\tau$ by

$$\widetilde{P}(A) = E\left[1_A \exp\left(-\int_0^\tau \boldsymbol{\gamma}(u)^\perp d\boldsymbol{B}_u - \frac{1}{2}\int_0^\tau |\boldsymbol{\gamma}(u)|^2\, du\right)\right]. \tag{5}$$

Specifically, if we set

$$r(t) = f(t,t) \quad \text{and} \quad \beta(t) = \exp\Big(\int_0^t r(u)\, du\Big), \tag{6}$$

then for each T the discounted process $\{P(t,T)/\beta(t) : 0 \le t \le T\}$ is a $\widetilde{P}$-martingale with respect to the filtration $\mathcal{F}_t$. The process $r(t) = f(t,t)$ is called the *spot rate* $r(t)$ and $\beta(t)$ is called the *accumulation factor* (or *discount factor*), and $\beta(t)$ is simply the value of a deposit that begins with a balance of one dollar at time zero and that accrues interest according to the stochastic spot rate $r(u)$ during the period $0 \le u \le t$.

The measure $\widetilde{P}$ is commonly called the *equivalent martingale measure* since $\widetilde{P}$ has the same null sets as P and since the process $\{P(t,T)/\beta(t) : 0 \le t \le T\}$ is a $\widetilde{P}$-martingale for each $T \in [0,\tau]$. The importance of $\widetilde{P}$ comes from the classic results of Harrison and Kreps (1979) and Harrison and Pliska (1981) that tell us that existence of such a measure is enough to guarantee that there are no arbitrage opportunities between the bonds of differing maturities.

A Basic Representation

By Girsanov's theorem and the definition (5) of $\widetilde{P}$, one sees that vector process defined by

$$\widetilde{\boldsymbol{B}}_t = \boldsymbol{B}_t + \int_0^t \boldsymbol{\gamma}(u)\, du \tag{7}$$

is a standard $\widetilde{P}$-Brownian motion, and Heath, Jarrow, and Morton (1992) observed that the SDE for $\{f(t,T)\}$ leads one to a particularly useful SDE for $\{P(t,T)\}$ Specifically, $\{P(t,T)\}$ satisfies the $\widetilde{\boldsymbol{B}}_t$ SDE

$$dP(t,T) = P(t,T)[\, r(t)\, dt + \boldsymbol{a}(t,T)^\perp d\widetilde{\boldsymbol{B}}_t\,], \tag{8}$$

where $\boldsymbol{a}(t,T)$ is the n-dimensional column vector of integrated volatilities defined by

$$\boldsymbol{a}(t,T) = -\int_t^T \boldsymbol{\sigma}(t,u)\, du. \tag{9}$$

The SDE (8) and the definition of $\beta(t)$ then permit one to show that for any *initial yield curve* $P(0,T)$ one has

$$P(t,T) = P(0,T)\beta(t) \exp\left[\int_0^t \boldsymbol{a}(s,T)^\perp d\widetilde{\boldsymbol{B}}_s - \frac{1}{2}\int_0^t |\boldsymbol{a}(s,T)|^2\, ds\right]. \tag{10}$$

A price process $\{P(t,T)\}$ that satisfies equations (1) through (10) is called an *HJM yield curve model*. Here we will find that the integral representation (10) gives one easy access to some basic qualitative features of the price process $\{P(t,T)\}$.

3. The Futures Rates — Forward Rates Inequality

Next, we need to recall some of the conventions and introduce some notation for LIBOR rates, specifically the "λ-LIBOR rate" that is offered at time t for a Eurodollar deposit for a maturity of $\lambda 360$ days. This rate is also called the *spot λ-LIBOR rate* when one needs to emphasize its distinction from the corresponding forward or futures rates and it is denoted by $L_\lambda(t)$.

By convention LIBOR rates are quote as *add-on yields*, and our first task here is to work out the relation between λ-LIBOR rates and the prices of the implied zero-coupon bonds. If $L_\lambda(t)$ denotes the λ-LIBOR rate at time t, then, in terms of a corresponding zero coupon bond, the arithmetic of add-on yields one finds the representation,

$$L_\lambda(t) = \frac{1}{\lambda}\left(\frac{1}{P(t,t+\lambda)} - 1\right) \qquad 0 < t < \tau, \tag{11}$$

and, in the same way, if $L_\lambda(t,T)$ denotes the *forward λ-LIBOR rate* for the future time period $[T, T+\lambda]$ one has the representation

$$L_\lambda(t,T) = \frac{1}{\lambda}\left(\frac{P(t,T)}{P(t,T+\lambda)} - 1\right) \qquad 0 < t < T < \tau. \tag{12}$$

The rate $L_\lambda(t,T)$ reflects the (add-on) interest rate available at time t for a riskless loan that begins at date T and which is paid back at time $T+\lambda$. The instantaneous forward rate $f(t,T)$ can be written in terms of the forward λ-LIBOR rate as $f(t,T) = \lim_{\lambda\downarrow 0} L_\lambda(t,T)$.

Now, if $\widetilde{E}$ denotes the expectation with respect to the equivalent martingale measure $\widetilde{P}$, then the conditional expectation under $\widetilde{E}$ can be used to provide a formula for the *λ-LIBOR futures rate* $F_\lambda(t,T)$. Specifically, we have the representation

$$F_\lambda(t,T) = \widetilde{E}\left[\frac{1}{\lambda}\left(\frac{1}{P(T,T+\lambda)} - 1\right)\Big|\mathcal{F}_t\right] = \widetilde{E}\left[L_\lambda(T)|\mathcal{F}_t\right] \quad 0 < t < T < \tau. \tag{13}$$

From the perspective of pure theory, one can take the formula (13) simply as the *definition* of the futures rate. Nevertheless, this formula is also widely considered to provide a sensible representation of real world futures prices, and there is a long history of modelling futures prices by such martingales. In particular, Karatzas and Shreve (1998, p. 43) provide a useful discussion of the economic motivation behind this definition.

Main Result: The Futures Rate-Forward Rate Inequality

Our main result is an inequality which asserts that the typical HJM framework yields a term structure that forces the λ-LIBOR futures rates $F_\lambda(t,T)$ to be higher than the associated forward λ-LIBOR rates $L_\lambda(t,T)$ *with probability one*. This bound is analogous to Theorem 1 of Pozdnyakov and Steele (2002), but here one meets two differences. First, we now deal with a more general volatility structure. Second, our analysis makes better used of the fact that the forward λ-LIBOR rate process $L_\lambda(\cdot,T)$ is a submartingale under the equivalent martingale measure.

Theorem 1 (Futures Rate-Forward Rate Inequality). *For all $0 < t < T < \tau$, the futures rate $F_\lambda(t,T)$ and the forward rate $L_\lambda(t,T)$ satisfy the inequality*

$$F_\lambda(t,T) \geq L_\lambda(t,T) \tag{14}$$

with probability one, provided that the underlying HJM family of bond prices $\{P(t,T) : 0 \leq t \leq T \leq \tau\}$ satisfies the constant sign condition *which asserts that for any t and $i = 1, 2, .., n$ the volatility coefficient $\sigma_i(t,\cdot)$ has the same sign.*

Proof. To begin, we will show that for any t, U, and T such that $0 \leq t \leq U \leq T \leq \tau$ we have a simple three-term product representation of the ratio $P(t,T)/P(t,U)$ any two bond prices :

$$\frac{P(t,T)}{P(t,U)} = \frac{P(0,T)}{P(0,U)}\eta(t,T,U)\xi(t,T,U), \tag{15}$$

where the second factor $\eta(t,T,U)$ is given by

$$\eta(t,T,U) = \exp\left[\int_0^t [\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)]^\perp \, d\widetilde{\boldsymbol{B}}_s - \frac{1}{2}\int_0^t |\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)|^2 \, ds\right],$$

and the third factor $\xi(t,T,U)$ is given by

$$\xi(t,T,U) = \exp\left[-\int_0^t \boldsymbol{a}(s,U)^\perp [\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)] \, ds\right].$$

To check this representation we just note that the basic bond formula (10) gives us

$$\begin{aligned}
\frac{P(t,T)}{P(t,U)} &= \frac{P(0,T)\beta(t)\exp\left[\int_0^t \boldsymbol{a}(s,T)^\perp d\widetilde{\boldsymbol{B}}_s - \frac{1}{2}\int_0^t |\boldsymbol{a}(s,T)|^2 \, ds\right]}{P(0,U)\beta(t)\exp\left[\int_0^t \boldsymbol{a}(s,U)^\perp d\widetilde{\boldsymbol{B}}_s - \frac{1}{2}\int_0^t |\boldsymbol{a}(s,U)|^2 \, ds\right]} \\
&= \frac{P(0,T)}{P(0,U)} \times \\
&\quad \times \exp\left[\int_0^t [\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)]^\perp \, d\widetilde{\boldsymbol{B}}_s - \frac{1}{2}\int_0^t [|\boldsymbol{a}(s,T)|^2 - |\boldsymbol{a}(s,U)|^2] \, ds\right] \\
&= \frac{P(0,T)}{P(0,U)}\eta(t,T,U)\xi(t,T,U).
\end{aligned}$$

The main benefit of this representation comes from the analytic properties of the last two factors.

The essential property of the third factor is that the process $\{\xi(s,T,U)\}$ is monotone decreasing as a function of s. To check this fact, we first note that by definition, the integrated volatility vector $\boldsymbol{a}(t,T)$ has components

$$a_i(t,T) = -\int_t^T \sigma_i(t,u) \, du,$$

so the constant sign condition for the values of $\boldsymbol{\sigma}(t,u)$, $t \leq u \leq U$ tells us that for each $s \in [0,U]$ and all $i = 1, 2, ..., n$, we have with probability one that

$$a_i(s,T) \leq a_i(s,U) \leq 0 \quad \text{or} \quad 0 \leq a_i(s,U) \leq a_i(s,T).$$

As a consequence, one finds that

$$-\boldsymbol{a}(s,U)^\perp [\boldsymbol{a}(s,T) - \boldsymbol{a}(s,U)] \leq 0, \quad \text{a.s.},$$

so by the integral representation of $\xi(s,T,U)$ one finds that it must be a decreasing function of s.

The essential property of the second factor $\{\eta(\cdot,T,U)\}$ is that it is a positive local $\widetilde{P}$-martingale, and, thus, by Fatou's lemma $\{\eta(\cdot,T,U)\}$ must also be a positive $\widetilde{P}$-supermartingale. Next, for any $0 \le U \le T \le \tau$ one then finds that the bond price ratio $P(\cdot,T)/P(\cdot,U)$ is the product of a positive $\widetilde{P}$-supermartingale $\eta(\cdot,T,U)$, a decreasing positive process $\xi(\cdot,T,U)$, and a nonnegative constant, so $P(\cdot,T)/P(\cdot,U)$ must itself be a *positive* $\widetilde{P}$-supermartingale.

Now, by Jensen's inequality, the convexity of $x \mapsto 1/x$ on $(0,\infty)$, and the supermartingale property of the bond price ratio $P(\cdot,T)/P(\cdot,U)$ for any choice of $0 \le t \le s \le U \le T \le \tau$, we have

$$\widetilde{E}\left(\frac{P(s,U)}{P(s,T)}\Big|\mathcal{F}_t\right) \ge \frac{1}{\widetilde{E}[P(s,T)/P(s,U)|\mathcal{F}_t]} \ge \frac{1}{P(t,T)/P(t,U)} = \frac{P(t,U)}{P(t,T)}.$$

The bottom line is that the reciprocal $P(\cdot,U)/P(\cdot,T)$ is a $\widetilde{P}$-submartingale.

Since the forward λ-LIBOR rate $L_\lambda(t,T)$ is a nonnegative affine function of the ratio process $P(\cdot,T)/P(\cdot,T+\lambda)$, we see that $\{L_\lambda(\cdot,T)\}$ is also a submartingale under the equivalent martingale measure $\widetilde{P}$. Finally, the λ-LIBOR futures rate $F_\lambda(\cdot,T)$ is a $\widetilde{P}$-martingale for which one has

$$F_\lambda(T,T) = L_\lambda(T) = L_\lambda(T,T),$$

so the submartingale property of the forward λ-LIBOR rate and the martingale property of λ-LIBOR futures rate together imply that

$$L_\lambda(t,T) \le \widetilde{E}[L_\lambda(T,T)|\mathcal{F}_t] = \widetilde{E}[L_\lambda(T)|\mathcal{F}_t] = F_\lambda(t,T) \quad \text{a.s.},$$

just as we intended to show. □

One should note that there are several methods that lead to a proof that the bond ratio $M_t = P(t,T)/P(t,U)$ is a $\widetilde{P}$-submartingale. The most immediate benefit of the present method may simply be that it is direct and self-contained, but there may also be benefits to be found in our introduction of the three term factorization (15). Certainly, the factorization provides more information than just the knowledge that $M_t = P(t,T)/P(t,U)$ is a $\widetilde{P}$-submartingale, although so far no specific use has been found for this additional information. Nevertheless, over time, one may expect that the factorization (15) will find a further role.

The Sign Condition

The only non-standard condition that one needs in order to obtain the futures rates-forward rates inequality is the constant sign condition, and one should note that this condition is met by most — but not all — of the specific HJM models that have been used in practice. For example, all of the examples in Heath, Jarrow, and Morton (1992) satisfy the constant sign condition, and in most cases the condition is trivial to check. In the continuous Ho–Lee model one has $\sigma(\omega,t,T) = \sigma$ where $\sigma > 0$ is constant, and in the Vasicek model one has $\sigma(\omega,t,T) = \sigma\exp(-\delta(T-t)) > 0$; moreover, the two-factor combinations of these models considered by Musiela and Rutkowski (1998, p. 324) satisfy the constant sign condition. One can also check that most of the models considered by Amin and Morton (1994) satisfy the constant sign condition, but some do not; for example, their "Linear Absolute" model with $\sigma(\omega,t,T) = \sigma_0 + \sigma_1(T-t)$ will not satisfy the constant sign condition for if σ_0 and σ_1 have opposite signs and T is sufficiently large.

4. The Swap Inequality

The most immediately useful consequence of the futures rate-forward rate inequality is that it quickly leads one to a theoretical upper bound for the swap rate. Moreover, this bound has the interesting and potentially important feature that it holds *uniformly* over a large class of the HJM models.

Before proving the swap rate inequality, we need to introduce some notation and recall some standard definitions. If $T_0 \geq 0$ and κ are given and we set

$$T_1 = T_0 + \lambda, T_2 = T_0 + 2\lambda, ..., T_N = T_0 + N\lambda,$$

then the *forward start payer swap settled in arrears* (or, in short, the *swap*) is a series of payments $\lambda\big[L_\lambda(T_k) - \kappa\big]$ that are made at the successive times T_{k+1} with $k = 0, ..., N-1$. In this payment formula, the constant κ is called the *pre-assigned fixed rate of interest*, and N is called the *length* of the swap. The time T_0 is called the *start date*, and, for the forward start payer swap settled in arrears, the times $T_0, ..., T_{N-1}$ are called the *reset dates* and the times $T_1, ..., T_N$ are called the *settlement dates.*

The time T_0 arbitrage price $\pi(\kappa)$ of the cash flows of a swap is given by

$$\pi(\kappa) = \sum_{k=0}^{N-1} \widetilde{E}\left[\frac{\beta(T_0)}{\beta(T_{k+1})}\lambda\big(L_\lambda(T_k) - \kappa\big)\big|\mathcal{F}_{T_0}\right],$$

and one can easily check that $\pi(\kappa)$ has the representation

$$\pi(\kappa) = 1 - \lambda\kappa[P(T_0, T_1) + \cdots + P(T_0, T_N)] - P(T_0, T_N),$$

a formula that is also derived and discussed in Musiela and Rutkowski (1997, p. 388). Finally, the *swap rate* κ_0 is such value of the preassigned fixed rate of interest κ that the time T_0 arbitrage price of the cash flows associated with the swap contract is zero, i.e.

$$\kappa_0 = \frac{1 - P(T_0, T_N)}{\lambda(P(T_0, T_1) + \cdots + P(T_0, T_N))}. \tag{16}$$

Now we are ready to present the main result of this section. This proof requires little more than seeing how the definition of the swap rate fits together with the futures rate inequality, but the resulting inequality still serves nicely when one tries to sort out the theoretical basis of the empirical observations of Burghardt and Hoskins (1995).

Theorem 2 (Swap Rate Inequality)**.** *Suppose $\{P(t,T) : 0 \leq t \leq T \leq \tau\}$ denotes an HJM family of bond prices for which the constant sign condition of Theorem 1 holds. The swap rate κ_0 then satisfies the following* swap rate inequality:

$$\kappa_0 \leq \frac{1 + \lambda L_\lambda(T_0) - \prod_{k=1}^{N-1}(1 + \lambda F_\lambda(T_0, T_k))^{-1}}{\lambda\big[1 + (1 + \lambda F_\lambda(T_0, T_1))^{-1} + \cdots + \prod_{k=1}^{N-1}(1 + \lambda F_\lambda(T_0, T_k))^{-1}\big]}. \tag{17}$$

Proof. First we need to develop a lower bound for the price of zero-coupon bond $P(T_0, T_k)$ with $k = 1, 2, ..., N$. Using the telescopic product and the formulas for the forward and spot LIBOR we have that

$$\begin{aligned} P(T_0, T_k) &= P(T_0, T_1)\frac{P(T_0, T_2)}{P(T_0, T_1)} \cdots \frac{P(T_0, T_k)}{P(T_0, T_{k-1})} \\ &= \frac{1}{1 + \lambda L_\lambda(T_0)} \frac{1}{1 + \lambda L_\lambda(T_0, T_1)} \cdots \frac{1}{1 + \lambda L_\lambda(T_0, T_{k-1})}. \end{aligned} \tag{18}$$

Now, if we replace the forward λ-LIBOR rates in this identity by the corresponding λ-LIBOR futures rates, then Theorem 1 tells us that the identity becomes an inequality,

$$P(T_0, T_k) \geq \frac{1}{1+\lambda L_\lambda(T_0)} \frac{1}{1+\lambda F_\lambda(T_0, T_1)} \cdots \frac{1}{1+\lambda F_\lambda(T_0, T_{k-1})}. \tag{19}$$

Finally, to obtain the swap rate inequality (17), we just need to substitute all the bond prices $P(T_0, T_N)$ in the formula for the swap rate (16) by their lower bounds from the bond inequality (19). When we the divide both the numerator and denominator of the resulting estimate by $(1+\lambda L_\lambda(T_0))^{-1}$, we see that the proof of the Swap Rate Inequality (17) is complete. $\square$

Surely the most interesting features of the Swap Rate Inequality is the fact that the expression on the right-hand side of (17) has often been used as an approximation to the true value of the swap rate κ_0. For example, the use of this approximation is expressly recommended in *Trading and Capital-Markets Activities Manual, Interest-Rate Swaps.*[1] Nevertheless, Theorem 2 suggests us that this widely used procedure may be subject to systematic biases. Specifically, in a world where the yield curve is well modelled by an HJM model that satisfies the constant volatility condition, we see from the Swap Rate Inequality (17) that the suggested approximation is *almost surely* an overestimate of the true swap rate. In a completely parallel way, the bond inequality (19) also shows that one faces an almost sure downward bias when one uses the expression on the right-hand side of (19) as an approximation for the price of zero-coupon bond. Nevertheless, as in Burghardt and Hoskins (1995) note (p. 63), such bond price approximations are also widely used.

5. Empirical Observations

One does not know *a priori* if the swap rate inequality (17) reflects a law of economic reality or if it is an artifact of the HJM model. One naturally wants to know if swap rate inequality (17) is evident in real-world swap rates. Fortunately, since July 3, 2000 the Federal Reserve Board has included the U.S. dollar par swap rates in the H.15 Daily Update,[2] so an empirical analysis of the swap rate inequality (17) can at least be begun.

In Figure 1, we provide a box plot of the value of the traditional swap rate approximation minus the observed swap rate (as derived from the FRB H.15 Daily Update). For example, in the first column of Figure 1, the top of the box marks the third quartile Q_3, the bottom marks the first quartile Q_1, the line interior to the box marks the median, and the value of the "upper fence" $U = Q_3 + 1.5(Q_3 - Q_1)$ and "lower fence" $L = Q_1 - 1.5(Q_3 - Q_1)$ are indicated by the square brackets. Observed differences beyond these fences are plotted individually as horizontal bars. Thus, if the 1-year swap rate on July 3, 2000 is 7.10% and the corresponding theoretical bound is $\approx$7.146%, then the plotted difference is .046%. Column one of Figure 1 summarizes the observed differences for the 1-year swap rates for each of the 247 trading days in the study period, and the remaining columns summarize the observed differences for the swap rates with maturities of 2,3,4,5 and 7 years.

[1] This document is publicly available on the website of the Board of Governors of the Federal Reserve System `www.bog.frb.fed.us/boarddocs/SupManual/` (see, e.g. p.7)

[2] see `www.federalreserve.gov/releases/H15`

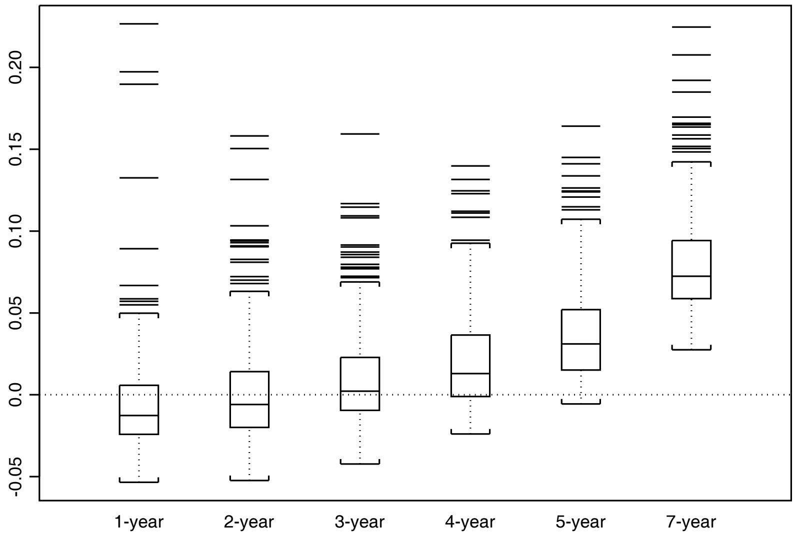

FIGURE 1. Box plots of the observed difference between the theoretical upper bounds and the observed swap rates for all trading days of the period July 2000 - June 2001

OBSERVATIONS ON MATURITIES OF 1 TO 7 YEARS

The data summarized by Figure 1 suggest several plausible inferences.

- For swaps rate with a relatively long time-to-maturity (4,5 and 7 years), one finds that the gap between the theoretical upper bound and the observed swap rate is by-and-large positive, just as theory would suggest.
- One also sees that the longer time-to-maturity the wider observed gap, and again this finding is consistent with our deductions under the HJM model.
- In contrast, one finds that for swaps with short maturities there are many days when the observed difference is negative, and this is at variance with the swap rate inequality (17) which predicts all of the observed differences should be positive. Ominously, the swaps with maturity of one year have a negative empirical gap almost 75% of the time.

Clearly the one-year swap rates deserve closer scrutiny, and in Figure 2 we provide for one-year swaps a month-by-month box plot for the gap between the theoretical upper bound on the swap rates and the observed swap rates.

OBSERVATIONS ON THE SHORT MATURITY SWAPS

The picture one draws from the data summarized in Figure 2 is much less compatible with the theoretical consequences of the HJM model. In particular, one finds:

- The violation of the theoretical upper bound are common for one-year swaps. In fact, for six of the months in the study period one finds that more than 75% of the observed differences were negative, while theory would predict that there would be no negative differences.

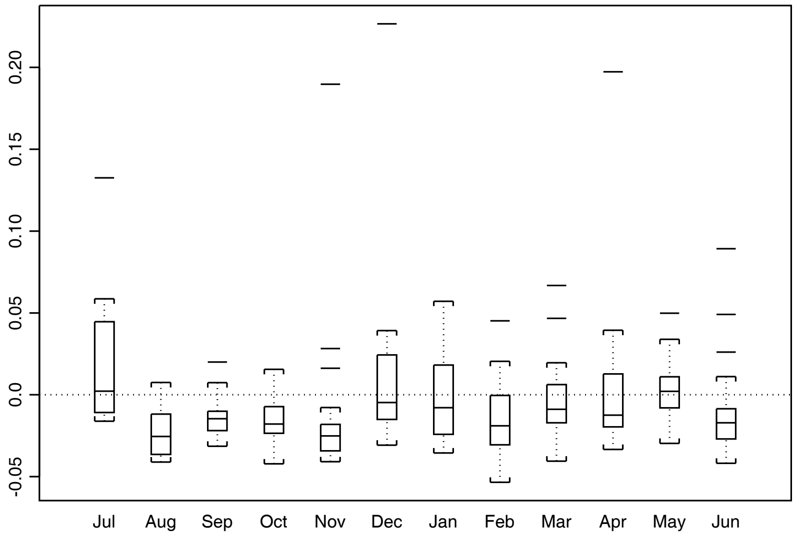

FIGURE 2. Box plots of the observed difference between the theoretical upper bounds and the observed 1-year swap rates month by month for the period July 2000 - June 2001

- The month of November 2000 was particularly extreme, and virtually all the observed differences are negative.
- The predominant gap size less than 5 basis points, but for the negative gaps one finds gap sizes that are a bit smaller. The violations of the theoretical bounds were typically less than 3 basis points.

One is hard pressed to say if gaps of the size observed here are economically significant, although the discussion of Burghardt and Hoskins (1995, p. 69) suggests that they may be. The more conservative conclusion that one might draw is that a constant sign volatility HJM model for futures rates may not be appropriate when the end goal is the pricing swap rates with short maturities. Even here one needs to be alert to possibility that the observed deficiencies may be remedied by more detailed models which take into account features such as transaction costs, counter party risk, or the fact that real-world futures are not continuously marked to market.

Data Resources and Computational Details

The Swap Rate Inequality (17) has three inputs: the swap rate κ_0, the spot LIBOR quote $L_\lambda(T_0)$, and the LIBOR futures rates $F_\lambda(T_0, T_k)$. Here T_0 denotes the current time and N is the length of the swap; so, for the 1-year swap linked to the 3-month LIBOR, one has $\lambda = .25$ and $N = 4$.

The only quantities in the Swap Rate Inequality that are not directly observed are the futures rates. Specifically, the Chicago Mercantile Exchange quotes the Eurodollar *futures prices*, not the *futures rates*, so one must make the obvious conversion (futures rate =1-futures price/100), but, unfortunately, there are two further problems with the CME data. The first is that one needs futures rates with the maturities that are not directly quoted, and the second is the more subtle problem that the rate quotes are not pegged to precisely the same times.

The swap rates published in the FRB H.15 Daily Update are based on information [3] obtained from acknowledged market makers as of 11:00 a.m. local time in New York, and one would surely like to have futures price quotes as of the same time as the swap rate survey. Unfortunately, the publicly available [4] data on Chicago Mercantile Exchange Eurodollar futures prices only cover the daily open, closing, highest and lowest futures prices. Rather than choosing to ignore the possible difference between Chicago opening prices and NYC 11:00 a.m. prices, we decided to report the observed gaps based on the *highest* futures rates. This choice was motivated by the desire to provide the most conservative estimate of the upper bound; since the right-hand side of the Swap Rate Inequality is an increasing function of the futures rates $f_\lambda(T_0, T_k)$, our use of the reported CME highest futures rate leads to a value that we can be sure to be an honest upper bound.

There were no further computational decisions of substance, but perhaps some small points should be recorded. We followed the tradition of using cubic splines to interpolate the term structure (as, for example, in Grinblatt and Jegadeesh (1996)), and we also used cubic splines to interpolate the term structure of the futures rates. For each day one has about 40 futures prices with maturities up to 10 years that can be used in order to construct an interpolated term structure of futures rates, so the interpolated rates should be rather reliable. The cubic spline interpolation was performed with the standard S-Plus function `spline`. The swap rate quotes provided in H.15 are stated on semiannual basis so the upper bounds were converted to annual rates by the usual transformation $x \mapsto (1 + x/2)^2 - 1$ after they were computed.

6. Concluding Remarks

The Futures Rate - Forward Rate Inequality (14) and the Swap Rate Inequality (17) were proved here under the assumption that the stochastic behavior of the yield curve can be specified by an HJM Model that satisfies the constant sign condition. Nevertheless, the phenomenon suggested by these inequalities is not necessarily restricted to such a model. The inequalities (14) and (17) can probably be stressed passed the breaking point if the constant sign condition is brutally violated, but we conjecture that for any economically feasible HJM model one will find that both inequalities will hold. Naturally, one difficult element of this conjecture is the embedded project of explaining just which of the HJM models are indeed economically realistic.

In fact, a richer question is whether there might be useful analogs to the Futures Rate - Forward Rate Inequality (14) or the Swap Rate Inequality (17) that hold in much greater generality, perhaps even for yield curve models that are *outside* of the HJM class of models. If the Futures Rate - Forward Rate Inequality (14) and the and the Swap Rate Inequality (17) do indeed reflect *bona fide* market realities,

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[3] see press release of ISDA `http://www.isda.org/press/a50398_1.html`

[4] These can be obtained from `www.barchart.com/cme/cmedta.htm`, as and other locations, and LIBOR quotes are most easily obtained from the British Bankers' Association `www.bba.org.uk`.

then one may well conjecture that there are analogs for these inequalities for a very wide range of yield curve models.

Acknowledgements: We are pleased to thank a referee for a suggestion that allowed us to remove an unnecessary regularity condition from Theorem 1 and to thank Krishna Ramaswamy for discussions on the intra-day non-contemporaneous pricing of swaps and futures between New York and Chicago.

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