Chapter 4. Bayesian Analysis (C5)

♠ Bayesian Robustness

♢ Posterior Robustness: Basic Concepts

• Basic Framework

Suppose a Bayesian is considering choice of an action, a, and is concerned about its robustness as π varies over a class Γ of priors. If $L(\theta, a)$ is the loss function and $\rho(\pi(\theta|\boldsymbol{x}), a)$ the posterior expected loss of a, it is natural to evaluate the robustness of a by considering

$$\left(\inf_{\pi \in \Gamma} \rho(\pi(\theta|\boldsymbol{x}), a), \sup_{\pi \in \Gamma} \rho(\pi(\theta|\boldsymbol{x}), a)\right);$$

this gives the range of possible posterior expected losses. Through the device of using inference losses, robustness can also be considered using this framework.

Example 1: Suppose the action, a, is to choose a credible set $C \subset \Theta$. Defining

$$L(\theta, C) = 1 - I_C(\theta),$$

so that

$$\rho(\pi(\theta|x), C) = 1 - P^{\pi(\theta|x)}(\theta \in C) = P^{\pi(\theta|x)}(\theta \notin C).$$

As a special example, suppose $X \sin N(\theta, 1)$,

$$\Gamma = \{\pi : \ \pi \text{ is } N(\mu, \tau^2),\ 1 \le \mu \le 2, 3 \le \tau^2 \le 4\},$$

and $x = 0$ is observed. Suppose the credible set $C = (-1, 2)$ is to be reported, and it is desired to determine tis minimum and maximum probabilities of containing θ as π ranges over Γ.

Note that, for $N(\mu, \tau^2)$ prior, the posterior is normal with mean and variance

$$\mu^\pi(x) = \frac{1}{(1+\tau^2)}\mu + \frac{\tau^2}{(1+\tau^2)}x = \frac{1}{(1+\tau^2)}\mu,$$

and

$$V^\pi = \frac{\tau^2}{1+\tau^2}.$$

Thus, the range of *posterior* corresponding to Γ is

$$\Gamma^*(x) = \Bigg\{\pi(\theta|0) : \pi(\theta|0) \text{ is } N\left(\tfrac{\mu}{1+\tau^2}, \tfrac{\tau^2}{1+\tau^2}\right), \\ 1 \le \mu \le 2, 3 \le \tau^2 \le 4\Bigg\}.$$

It follows that

$$P^{\pi(\theta|0)}((-1,2)) = \int_{-1}^{2} \pi(\theta|0)d\theta$$

$$=\Phi\left(\frac{2-\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right) - \Phi\left(\frac{-1-\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right)$$

$$=\Phi\left(\frac{2-\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right) + \Phi\left(\frac{1+\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right) - 1,$$

where $\Phi(\cdot)$ is the $N(0,1)$ cdf. Now,

$$\frac{\partial P^{\pi(\theta|0)}((-1,2))}{\partial \mu}$$

$$=\frac{1}{\sqrt{\tau^2(1+\tau^2)}}\left[-\phi\left(\frac{2-\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right) + \phi\left(\frac{1+\frac{\mu}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right)\right] > 0,$$

for $1 \le \mu \le 2$ and $3 \le \tau^2 \le 4$, where $\phi(\cdot)$ is the

$N(0,1)$ pdf. Note that in this case,

$$2+2\tau^2-\mu \geq 1+\tau^2+\mu \iff 1+\tau^2 \geq 2\mu.$$

Thus, $P^{\pi(\theta|0)}((-1,2))$ is an increasing function of μ.

For $\mu=2$,

$$\begin{aligned}&P^{\pi(\theta|0)}((-1,2))\\ =&\Phi\left(\frac{2-\frac{2}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right)+\Phi\left(\frac{1+\frac{2}{1+\tau^2}}{\sqrt{\frac{\tau^2}{1+\tau^2}}}\right)-1,\end{aligned}$$

Let $\delta=\sqrt{\frac{\tau^2}{1+\tau^2}}$. Then

$$\sqrt{0.75} \leq \delta \leq \sqrt{0.8},$$

$$P^{\pi(\theta|0)}((-1,2))=\Phi(2\delta)+\Phi\left(\frac{3}{\delta}-2\delta\right)-1,$$

and

$$\frac{\partial P^{\pi(\theta|0)}((-1,2))}{\partial\delta}=2\phi(2\delta)-(3/\delta^2+2)\phi\left(\frac{3}{\delta}-2\delta\right).$$

Since $2 < 3/\delta^2 + 2$ and

$$\frac{3}{\delta} - 2\delta \leq 2\delta \iff 0.75 = (3/4) \leq \delta^2,$$

$$\frac{\partial P^{\pi(\theta|0)}((-1,2))}{\partial \delta} \leq 0.$$

Therefore, $P^{\pi(\theta|0)}((-1,2))$ is a decreasing function of δ, and the maximum of $P^{\pi(\theta|0)}((-1,2))$ is attained at $\delta = \sqrt{0.75}$. Plugging $\delta = \sqrt{0.75}$ into $P^{\pi(\theta|0)}((-1,2))$, we obtain

$$\begin{aligned} P^{\pi(\theta|0)}((-1,2)) =& \Phi(2\sqrt{0.75}) + \Phi\left(\frac{3}{\sqrt{0.75}} - 2\sqrt{0.75}\right) - 1 \\ =& 2\Phi(1.73) - 1 = 2(0.9582) - 1 = 0.916. \end{aligned}$$

Similarly, we can show that when $\mu = 1$, $P(^{\pi(\theta|0)}((-1,2))$ is minimized at $\tau^2 = 4$ and the minimum value of $P(^{\pi(\theta|0)}((-1,2))$ is 0.888. Thus,

$$0.888 \leq P(^{\pi(\theta|0)}(C) \leq 0.916$$

or

$$0.084 \leq \rho(\pi(\theta|0), C) \leq 0.112.$$

Definition: The Γ-*posterior expected loss of* a_0 is

$$\rho_\Gamma(a_0) = \sum_{\pi \in \Gamma} \rho(\pi(\theta|\boldsymbol{x}), a_0).$$

Example 1 (continued): We calculated that

$$\rho_\Gamma(C) = 0.112.$$

This corresponds to saying that C is *at least* an 88.8% credible set.

Definition: An action a_0 is ϵ-*posterior robust* with respect to Γ if, for all $\pi \in \Gamma$,

$$|\rho(\pi(\theta|\boldsymbol{x}), a_0) - \inf_a \rho(\pi(\theta|\boldsymbol{x}), a)| \leq \epsilon.$$

Example 2: Suppose it is desired to estimate θ under $L(\theta, a) = (\theta - a)^2$. We have

$$\rho(\pi(\theta|\boldsymbol{x}), a_0) = V^\pi(\boldsymbol{x}) + (\mu^\pi - a_0)^2,$$

where μ^π and V^π are the posterior mean and variance. Also, we know that the minimum of $\rho(\pi(\theta|\boldsymbol{x}), a)$ is achieved at $a = \mu^\pi(\boldsymbol{x})$, which has posterior expected loss $V^\pi(\boldsymbol{x})$. Hence

$$|\rho(\pi(\theta|\boldsymbol{x}), a_0) - \inf_a \rho(\pi(\theta|\boldsymbol{x}), a)| = (\,u^\pi(\boldsymbol{x}) - a_0)^2.$$

Thus a_0 is ϵ-posterior robust if it is within $\pm\sqrt{\epsilon}$ of all the posterior means corresponding to priors in Γ. In Example 1, the posterior means $(\mu/(1+\tau^2))$ range over the interval $[0.2, 0.5]$, so $a_0 = 0.35$ would be $(0.15)^2 = 0.0225$-posterior robust.

♢ Posterior Robustness: ϵ-Contamination Class

The ϵ-contamination class of priors, defined by

$$\Gamma = \{\pi : \pi = (1-\epsilon)\pi_0 + \epsilon q,\ q \in \mathcal{Q}\},$$

is particularly attractive to work with when investigating posterior robustness.

Lemma 1: *Suppose* $\pi = (1-\epsilon)\pi_0 + \epsilon q$, *that the posterior densities* $\pi_0(\theta|\boldsymbol{x})$ *and* $q(\theta|\boldsymbol{x})$ *exist, and that* $m(\boldsymbol{x}|\pi) > 0$. *Then*

$$\pi(\theta|\boldsymbol{x}) = \lambda(\boldsymbol{x})\pi_0(\theta|\boldsymbol{x}) + [1-\lambda(\boldsymbol{x})]q(\theta|\boldsymbol{x}),$$

where

$$\lambda(\boldsymbol{x}) = \frac{(1-\epsilon)m(\boldsymbol{x}|\pi_0)}{m(\boldsymbol{x}|\pi)} = \left[1 + \frac{\epsilon m(\boldsymbol{x}|q)}{(1-\epsilon)m(\boldsymbol{x}|\pi_0)}\right]^{-1},$$

and $m(\boldsymbol{x}|\pi) = (1-\epsilon)m(\boldsymbol{x}|\pi_0) + \epsilon m(\boldsymbol{x}|q)$.
Furthermore, in a decision problem,

$$\begin{aligned}\rho(\pi(\theta|\boldsymbol{x}), a) &= E^{\pi(\theta|\boldsymbol{x})}[L(\theta, a)]\\ &= \lambda(\boldsymbol{x})\rho(\pi_0(\theta|\boldsymbol{x}), a) + [1-\lambda(\boldsymbol{x})]\rho(q(\theta|\boldsymbol{x}), a).\end{aligned}$$

Example 3: To find the posterior mean, $\mu^\pi(\boldsymbol{x})$, for $\pi \in \Gamma$, set $L(\theta, a) \equiv \theta$, yielding

$$\begin{aligned}\mu^\pi(\boldsymbol{x}) =&\lambda(\boldsymbol{x})E^{\pi_0(\theta|\boldsymbol{x})}[\theta] + [1-\lambda(\boldsymbol{x})]E^{q(\theta|\boldsymbol{x})}[\theta] \\ =&\lambda(\boldsymbol{x})\mu^{\pi_0}(\boldsymbol{x}) + [1-\lambda(\boldsymbol{x})]\mu^q(\boldsymbol{x}).\end{aligned}$$

To find the posterior variance, $V^\pi(\boldsymbol{x})$, set $L(\theta, a) \equiv (\theta - \mu^\pi(\boldsymbol{x}))^2$, yielding

$$\begin{aligned}&V^\pi(\boldsymbol{x}) \\ =&\lambda(\boldsymbol{x})E^{\pi_0(\theta|\boldsymbol{x})}[(\theta-\mu^\pi)^2] + [1-\lambda(\boldsymbol{x})]E^{q(\theta|\boldsymbol{x})}[(\theta-\mu^\pi)^2] \\ =&\lambda(\boldsymbol{x})[V^{\pi_0} + (\mu^{\pi_0}-\mu^\pi)^2] + [1-\lambda(\boldsymbol{x})][V^q + (\mu^q-\mu^\pi)^2] \\ =&\lambda(\boldsymbol{x})V^{\pi_0}(\boldsymbol{x}) + [1-\lambda(\boldsymbol{x})]V^q(\boldsymbol{x}) \\ &+ \lambda(\boldsymbol{x})[1-\lambda(\boldsymbol{x})][\mu^{\pi_0}(\boldsymbol{x}) - \mu^q(\boldsymbol{x})]^2.\end{aligned}$$

Theorem 1: *Suppose* $\mathcal{Q} = \{$ *all distribution* $\}$ *and* $L(\theta, a) = I_C(\theta)$*, so that*

$$\rho(\pi(\theta|\boldsymbol{x}), a) = P^{\pi(\theta|\boldsymbol{x})}(\theta \in C).$$

Then

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C) = P_0 \times \left[1 + \frac{\epsilon \sup_{\theta \notin C} f(\boldsymbol{x}|\theta)}{(1-\epsilon) m(\boldsymbol{x}|\pi_0)}\right]^{-1},$$

and

$$\begin{aligned} &\sup_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C) \\ =&1 - (1 - P_0) \left[1 + \frac{\epsilon \sup_{\theta \in C} f(\boldsymbol{x}|\theta)}{(1-\epsilon) m(\boldsymbol{x}|\pi_0)}\right]^{-1}, \end{aligned}$$

where $P_0 = P^{\pi_0(\theta|\boldsymbol{x})}(\theta \in C)$.

Proof: The formula for $\sup_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$ directly follows from the formula for $\inf_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$. Thus, it is sufficient to show

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C) = P_0 \times \left[1 + \frac{\epsilon \sup_{\theta \notin C} f(\boldsymbol{x}|\theta)}{(1-\epsilon)m(\boldsymbol{x}|\pi_0)}\right]^{-1}.$$

Let $\overline{C}$ denote the complement of C. Also, for any $q \in \mathcal{Q}$, let

$$z_q(A) = \int_A f(\boldsymbol{x}|\theta)q(d\theta).$$

Clearly

$$P^{\pi(\theta|\boldsymbol{x})}(\theta \in C) = \frac{(1-\epsilon)m(\boldsymbol{x}|\pi_0)P_0 + \epsilon z_q(C)}{(1-\epsilon)m(\boldsymbol{x}|\pi_0) + \epsilon z_q(C) + \epsilon z_q(\overline{C})}.$$

Consider the function

$$h(z) = \frac{K_1 + z}{K_2 + z + g(z)}.$$

It is straightforward to check that h is increasing in $z \geq 0$ when $K_2 \geq K_1 \geq 0$ and q is a positive, decreasing function of z. Setting

$$K_1 = (1-\epsilon)m(\boldsymbol{x}|\pi_0)P_0 \;\text{ and }\; K - 2 = (1-\epsilon)m(\boldsymbol{x}|\pi_0),$$

it follows that $P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$ can be decreased by taking any mass that q assigns to C and giving it to

$\overline{C}$; thus $P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$ is minimized when

$$z_q(C) = 0.$$

Furthermore,

$$\begin{aligned}
&\inf_{\pi\in\Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C) \\
&= \inf_{\{q:\ z_q(C)=0\}} \frac{(1-\epsilon)m(\boldsymbol{x}|\pi_0)P_0}{(1-\epsilon)m(\boldsymbol{x}|\pi_0) + \epsilon z_q(\overline{C})} \\
&= \frac{(1-\epsilon)m(\boldsymbol{x}|\pi_0)P_0}{(1-\epsilon)m(\boldsymbol{x}|\pi_0) + \epsilon \sup_{\{q:\ z_q(C)=0\}} z_q(\overline{C})}.
\end{aligned}$$

But

$$\sup_{\{q:\ z_q(C)=0\}} z_q(\overline{C}) = \sup_{\theta\in\overline{C}} f(\boldsymbol{x}|\theta),$$

which proves the theorem. $\square$

• **Note**: The major applications of Theorem 1 are to robustness of credible sets (where C is the credible set), and to hypothesis testing (where C defines a hypothesis).

Example 4: Suppose that $X \sim N(\theta, \sigma^2)$ and that $\pi_0(\theta)$ is $N(\mu, \tau^2)$. We know that the $100(1-\alpha)\%$ HPD credible set for θ under π_0 is

$$C = \left(\mu^\pi(x) + z\left(\frac{\alpha}{2}\right)\sqrt{V^\pi(x)},\ \mu^\pi(x) - z\left(\frac{\alpha}{2}\right)\sqrt{V^\pi(x)}\right),$$

where

$$\mu^\pi(x) = x - \left[\frac{\sigma^2}{\sigma^2+\tau^2}\right](x-\mu)$$

and

$$V^\pi(x) = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}.$$

To investigate the robustness of C over the ϵ-contamination class of priors, with $\mathcal{Q}$={all distributions}, we use Theorem 1.

Note first that, if $x \in C$, then

$$\sum_{\theta \in C} f(x|\theta) = f(x|x) = (2\pi\sigma^2)^{-1/2},$$

while

$$\sum_{\theta \notin C} f(x|\theta) = f(x \mid \text{endpoint of } C \text{ closest to } x)$$

$$=(2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(\text{endpoint} - x)^2\right\}$$

$$=(2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}\left(|\mu^\pi(x) - x| + z\left(\frac{\alpha}{2}\right)\sqrt{V^\pi}\right)^2\right\}.$$

Since $P_0 = 1 - \alpha$ and

$$m(x|\pi_0) = [2\pi(\sigma^2 + \tau^2)]^{-1/2} \exp\left\{-\frac{(x-\mu(^2}{2(\sigma^2+\tau^2)}\right\},$$

we have determined all quantities in $\inf_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$ and $\sup_{\pi \in \Gamma} P^{\pi(\theta|\boldsymbol{x})}(\theta \in C)$. If $x \notin C$, then the right-hand sides of $\sum_{\theta \in C} f(x|\theta)$ and $\sum_{\theta \notin C} f(x|\theta)$ should be interchanged.

As a concrete example, suppose that $\sigma^2 = 1$, $\tau^2 = 2$, $\mu = 0$, and $\epsilon = 0.1$. First, suppose $x = 1$ is observed. Then $\mu^\pi(1) = \frac{2}{3}$, $V^\pi = \frac{2}{3}$, and the 95% HPD interval is

$$C = \frac{2}{3} \pm (1.96)\left(\frac{2}{3}\right)^{1/2} = (-0.93, 2.27),$$

so that $x = 1 \in C$. Hence we have

$$\sup_{\theta \in C} f(1|\theta) = (2\pi)^{-1} = 0.4$$

and

$$\sup_{\theta \notin C} f(1|\theta) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(|\frac{2}{3} - 1| - 1.96(\frac{2}{3})^{1/2})^2\} = 0.18.$$

Also, $P_0 = 1 - \alpha = 0.95$ and $m(1|\pi_0) = 0.19$, so that Theorem 1 yields

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|1)}(\theta \in C) = 0.95\left[1 + \frac{0.1(0.18)}{0.9(0.19)}\right]^{-1} = 0.86,$$

$$\sup_{\pi \in \Gamma} P^{\pi(\theta|1)}(\theta \in C) = 1-(1-0.95)\left[1+\frac{0.1(0.4)}{0.9(0.19)}\right]^{-1} = 0.96.$$

Thus our "confidence" in C actually ranges between 0.86 and 0.96 as π ranges over Γ.

Suppose, instead, that $x = 3$ is observed. Then $\mu^\pi(3) = 2$ and $C = (0.4, 3.6)$. Again, $3 \in C$, so we calculate

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|3)}(\theta \in C) = 0.95\left[1 + \frac{0.1(0.33)}{0.9(0.051)}\right]^{-1} = 0.55,$$

$$\sup_{\pi \in \Gamma} P^{\pi(\theta|3)}(\theta \in C) = 1-(1-0.95)\left[1+\frac{0.1(0.4)}{0.9(0.051)}\right]^{-1} = 0.97.$$

Example 5: Again suppose that $X \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, but now we desire to test H_0: $\theta \le \theta_0$ versus H_1: $\theta > \theta_0$. Defining $C = (-\infty, \theta_0)$, we have that the posterior probability of H_0 is simply that of C. The formulas in Theorem 1 and Example 4 thus apply directly to this problem.

As a concrete example, suppose $\sigma^2 = 1$, $\mu = 0$, $\tau^2 = 2$, $\theta_0 = 0$, and $x = 2$ is observed. Then the posterior probability of C under π_0 is

$$P_0 = P^{\pi_0(\theta|2)}(C) = \Phi([0-\mu^\pi(2)]/\sqrt{V^\pi}) = \Phi(-1.63) = 0.052.$$

Note that $x = 2 \notin C$, so that we calculate

$$\sup_{\theta \in C} f(2|\theta) = (2\pi)^{-1/2}\exp\{-\frac{1}{2}(0-2)^2\} = 0.054$$

and

$$\sup_{\theta \notin C} f(2|\theta) = (2\pi)^{-1/2} = 0.40.$$

Also $m(2|\pi_0) = 0.12$. Theorem 1 yields

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|2)}(\theta \in C) = 0.052\left[1 + \frac{0.1(0.40)}{0.9(0.12)}\right]^{-1} = 0.038,$$

and

$$\sup_{\pi \in \Gamma} P^{\pi(\theta|2)}(\theta \in C)$$

$$=1 - (1 - 0.052)\left[1 + \frac{0.1(0.054)}{0.9(0.12)}\right]^{-1} = 0.097.$$

• **Note 1**: An important feature of posterior robustness is that it is generally highly dependent on the x observed. Thus, for $x = 2$ in Example 4, the nominal 95% set maintains reasonably high probability for all $\pi \in \Gamma$. For $x = 3$, however, the probability of the nominal 95% credible set can drop to as low as 0.55 for all $\pi \in \Gamma$.

• **Note 2**: What is to be done when posterior robustness is found to be lacking as π varies over Γ? The first consideration should be to determine if the lack of posterior robustness is due to Γ containing unreasonable prior distributions. It will often be the case that $\mathcal{Q} = \{$ all distributions $\}$ is too large. A relatively simple alternative to using the "too large" Γ is to use the ϵ-contamination class, with $\mathcal{Q}$ chosen to be a parametric class of distributions.

Example 6: Suppose $X \sim N(\theta, \sigma^2)$ and π_0 is $N(\mu, \tau^2)$. Consider the ϵ-contamination class, Γ, with

$$\mathcal{Q} = \{q_k : \quad q_k \text{ is } \quad \mathcal{U}_k = (\mu - k, \mu + k), k > 0\}.$$

To find the range of posterior probabilities of an interval $C = (c_1, c_2)$ as π ranges over Γ, we need to consider

$$P^{\pi(\theta|x)}(\theta \in C) = \lambda_k(x) P_0 + (1 - \lambda_k(x)) Q_k,$$

where

$$P_0 = P^{\pi_0(\theta|x)}(\theta \in C),$$

$$\lambda_k(x) = \left[1 + \frac{\epsilon}{1-\epsilon} \cdot \frac{m(x|q_k)}{m(x|\pi_0)}\right]^{-1},$$

$$\begin{aligned} m(x|q_k) &= \int_{\mu-k}^{\mu+k} f(x|\theta) \cdot \frac{1}{2k} d\theta \\ &= \frac{1}{2k}\left[\Phi\left(\frac{\mu+k-x}{\sigma}\right) - \Phi\left(\frac{\mu-k-x}{\sigma}\right)\right], \end{aligned}$$

$$\begin{aligned} Q_k(x) =& P^{q_k(\theta|x)}(\theta \in C) = \frac{1}{m(x|q_k)} \int_{c^*}^{c^{**}} f(x|\theta) \cdot \frac{1}{2k} d\theta \\ =& \frac{1}{2km(x|q_k)} \left[\Phi\left(\frac{c^{**}-x}{\sigma}\right) - \Phi\left(\frac{c^*-x}{\sigma}\right)\right]^+, \end{aligned}$$

$$c^* = \max\{c_1, -k\}, \quad c^{**} = \max\{c_2, k\},$$

and "+" stands for the positive part.

As a concrete example, suppose $\epsilon = 0.1$, $\sigma^2 = 1$, $\mu = 0$, $\tau^2 = 2$, $x = 1$, and $C = (-0.93, 2.27)$. Then numerical calculation gives

$$\inf_{\pi \in \Gamma} P^{\pi(\theta|x)}(\theta \in C) = 0.945 \text{ (achieved at } k = 3.4),$$

$$\sup_{\pi \in \Gamma} P^{\pi(\theta|x)}(\theta \in C) = 0.956 \text{ (achieved at } k = 0.93).$$

Thus we have excellent posterior robustness.

For the case $x = 3$ with $C = (0.4, 3.6)$, the inf is 0.913 (achieved at $k = 5.2$) and the sup is 0.958 (achieved at $k = 3.6$). Recalling that the corresponding inf in Example 4 was 0.55, it seems clear that the apparent nonrobustness in that example ($x = 3$) was due to the unreasonable Γ. It would thus seem quite safe to call C at least a 90% credible set.