DIFFERENTIAL GEOMETRY OF $ARFIMA$ PROCESSES

Nalini Ravishanker

Department of Statistics
University of Connecticut
U-3120, 196 Auditorium Road
Storrs, CT 06269

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ABSTRACT

Autoregressive fractionally integrated moving average ($ARFIMA$) processes are widely used for modeling time series exhibiting both long-memory and short-memory behavior. Properties of Toeplitz matrices associated with the spectral density functions of Gaussian $ARFIMA$ processes are used to compute differential geometric quantities.

1. INTRODUCTION

Time series data occurring in several areas such as geology, hydrology and economics exhibit both short-memory and long-memory behavior, which may be modeled by the class of autoregressive fractionally integrated moving average ($ARFIMA$) processes (Beran, 1994). A time series $\{X_t\}$ is generated by an autoregressive fractionally integrated moving average ($ARFIMA$) process if

$$\phi(B)(1-B)^d X_t = \theta(B)\varepsilon_t, \tag{1}$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \cdots \theta_q B^q$ are polynomials in B of degrees p and q respectively, p and q are integers, B is the backshift operator, i.e. $BX_t = X_{t-1}$, d is a real number denoting the fractional degree of differencing, the fractional difference operator is defined by a binomial series

$$(1-B)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-B)^j$$

and the ε_t are independent and identically distributed as normal random variables with mean 0 and variance σ^2. It is assumed that $-\frac{1}{2} < d < \frac{1}{2}$ and that the roots of $\phi(z) = 0$ and $\theta(z) = 0$ lie outside the unit circle, ensuring the stationarity and invertibility of the process. It is further assumed that $\phi(z)$ and $\theta(z)$ do not have common roots. Let $\boldsymbol{\beta} = (\phi_1, ..., \phi_p, \theta_1, ..., \theta_q, \sigma^2, d)$ denote the vector of model parameters of dimension $m = p + q + 2$. Since the effect of the parameter d on distant observations decays hyperbolically as a function of increasing lags while the effect of the ϕ and θ parameters decays exponentially, (1) is useful in modeling time series that exhibit both short-memory and long-memory behavior. When $p \neq 0$, $q \neq 0$ and $d = 0$, (1) reduces to the $ARMA(p, q)$ process; when $p = 0$, $q = 0$ and $d \neq 0$, (1) represents the fractional Gaussian noise

$$(1-B)^d X_t = \varepsilon_t. \tag{2}$$

Based on a sample of n observations $\mathbf{X}_n = (X_1, ..., X_n)'$ generated by an $ARFIMA(p, d, q)$ process (1), the exact likelihood function has the form

$$L(\mathbf{X}_n; \boldsymbol{\beta}) = (2\pi)^{-n/2} \mid \Sigma \mid^{-\frac{1}{2}} \exp\{-\frac{1}{2}\mathbf{X}_n' \Sigma^{-1} \mathbf{X}_n\}, \tag{3}$$

where $\Sigma = Cov(\mathbf{X}_n)$ and is a function of $\boldsymbol{\beta}$ with elements given by γ_k^x, the autocovariances of $\{X_t\}$ of lag k. The spectral density function of $\{X_t\}$ has the form

$$s_{\boldsymbol{\beta}}(w) = \frac{\sigma^2}{2\pi}\frac{\theta(e^{\mathrm{i}w})\theta(e^{-\mathrm{i}w})}{\phi(e^{\mathrm{i}w})\phi(e^{-\mathrm{i}w})}\{(1-e^{\mathrm{i}w})(1-e^{-\mathrm{i}w})\}^{-d},\ 0 < w \leq 2\pi, \tag{4}$$

and i$=\sqrt{-1}$. While the spectral density of the short-memory $ARMA(p,q)$ process is a bounded rational function, $s_{\boldsymbol{\beta}}(w)$ is unbounded at $w=0$.

Properties such as consistency, first-order asymptotic efficiency, and asymptotic normality of the approximate and exact MLEs of the model parameters have been discussed in Fox and Taqqu (1986) and Dahlhaus (1989). The study of higher-order asymptotic inference for the $ARFIMA(p,d,q)$ process is of considerable interest and can be related to the differential geometry of the process. The use of differential geometry to characterize statistical inference has been widely studied in the last two decades (Barndorff-Nielsen et.al., 1986, Murray and Rice, 1993). The relationship between the geometry and asymptotic inference in statistical models was pointed out by Rao (1961, 1962, 1963). Application of the geometrical approach for time series problems was addressed by Amari (1987) who discussed the geometrical theory of manifolds for linear systems, and by Ravishanker et.al. (1990), where the geometry of $ARMA(p,q)$ processes was discussed. The latter characterized the $ARMA(p,q)$ processes as members of the curved exponential family, derived their geometric properties and used them to obtain for some simple processes (a) approximate higher-order asymptotic bias of the maximum likelihood estimators of parameters, (b) parameter transformations to satisfy predetermined statistical properties and (c) the Bartlett correction to the likelihood ratio test statistic.

In this paper, analytical expressions for various geometric properties in $ARFIMA(p,d,q)$ processes are computed by utilizing results on Toeplitz matrices associated with their spectral density functions (Dahlhaus, 1989). Due to the unboundedness of $s_{\boldsymbol{\beta}}(w)$ at $w=0$, these computations are considerably different from those used with $ARMA$ processes. A brief review of the geometry is given in Section 2 and the computational procedure is described in Section 3, where the geometric quantities are presented in detail. Section 3 also shows how these quantities may be used to obtain affine parametrizations for the fractional Gaussian noise parameter. Section 4 suggests application of these geometrical quantities for carrying out asymptotic inference.

2. A REVIEW OF DIFFERENTIAL GEOMETRY

Let $\mathcal{S}$ denote distributions from an r-parameter exponential family defined on a sample space $\mathcal{X}$ and let $\boldsymbol{\eta}$ denote the r-dimensional canonical parameter of the family. Based on a set of n observations $\mathbf{x} = (x_1, ..., x_n)'$ from this family, the log-likelihood function is

$$\ell(\mathbf{x}; \boldsymbol{\eta}) = \boldsymbol{\eta}'\mathbf{t}(\mathbf{x}) - \psi(\boldsymbol{\eta}) + c(\mathbf{x}),$$

where $\mathbf{t}(\mathbf{x})$ is the r-dimensional vector of minimal sufficient statistics, $c(\mathbf{x})$ is a function defined on $\mathcal{X}$, $\psi(\boldsymbol{\eta}) = \log \int \exp\{\boldsymbol{\eta}'\mathbf{t}(\mathbf{x}) + c(\mathbf{x})\} d\mu$ and μ is a measure over $\mathcal{X}$. An m-dimensional curved exponential family $\mathcal{M}$ parametrized by $\boldsymbol{\beta}$ is a subfamily of $\mathcal{S}$ and is specified by expressing the canonical parameter $\boldsymbol{\eta}$ as a smooth function $\boldsymbol{\eta}(\boldsymbol{\beta})$ where $\dim(\boldsymbol{\beta}) = m < r$. $\mathcal{M}$ is then referred to as an $(r, m)-$curved exponential family; in a curved exponential family the dimension of the minimal sufficient statistic exceeds the dimension of the parameter space (Efron, 1975). The families $\mathcal{S}$ and $\mathcal{M}$ may be considered as differentiable manifolds with respective coordinate systems $\boldsymbol{\eta}$ and $\boldsymbol{\beta}$, and $\mathcal{M}$ is a submanifold of $\mathcal{S}$ (Amari, 1985). The Riemannian metric tensor and the α-connections in $\mathcal{S}$ are defined respectively by $g_{ij}(\boldsymbol{\eta})$ and $\overset{\alpha}{\Gamma}_{ijk} (\boldsymbol{\eta})$ where

$$g_{ij}(\boldsymbol{\eta}) = E(\ell_i \ell_j), (i, j = 1, ..., r), \tag{5}$$

$$\overset{\alpha}{\Gamma}_{ijk} (\boldsymbol{\eta}) = E(\ell_{ij}\ell_k + \frac{(1-\alpha)}{2}\ell_i\ell_j\ell_k), (i, j, k = 1, ..., r), \tag{6}$$

where $\ell_i = \frac{\partial \ell}{\partial \eta_i}(\mathbf{x};\boldsymbol{\eta})$, $\ell_{ij} = \frac{\partial^2 \ell}{\partial \eta_i \partial \eta_j}(\mathbf{x};\boldsymbol{\eta})$ and α is a real number. Note that $g_{ij}(\boldsymbol{\eta})$ corresponds to the $(i,j)^{th}$ element of the expected Fisher information matrix for $\boldsymbol{\eta}$ and the α-connections $\overset{\alpha}{\Gamma}_{ijk}(\boldsymbol{\eta})$ are affine connections that allow the comparison of vectors in two tangent spaces of $\mathcal{X}$ at neighboring points $\boldsymbol{\eta}$ and $\boldsymbol{\eta} + \mathbf{d}\boldsymbol{\eta}$. Let $\boldsymbol{\eta}^*$ denote an alternate coordinate system in $\mathcal{S}$; the corresponding Riemannian metric tensor $g_{IJ}(\boldsymbol{\eta}^*)$ and α-connections $\overset{\alpha}{\Gamma}_{IJK}(\boldsymbol{\eta}^*)$ may be expressed in terms of the original coordinate system $\boldsymbol{\eta}$ by means of the coordinate transformation laws

$$g_{IJ}(\boldsymbol{\eta}^*) = B_I^i B_J^j g_{ij}(\boldsymbol{\eta}) \tag{7}$$

and

$$\overset{\alpha}{\Gamma}_{IJK}(\boldsymbol{\eta}^*) = B_I^i B_J^j B_K^k \overset{\alpha}{\Gamma}_{ijk}(\boldsymbol{\eta}) + g_{jk}(\boldsymbol{\eta}) B_K^k \partial_I B_J^j, \tag{8}$$

where $B_I^i = \frac{\partial \eta_i}{\partial \eta_I^*}$, $\partial_I B_J^j = \frac{\partial^2 \eta_j}{\partial \eta_I^* \partial \eta_J^*}$, $i,j,k = 1,...,r$; $I,J,K = 1,...,r$. Note that Einstein's summation convention is used in the above equations, and in the rest of the paper, i.e. summation is assumed over indices which appear both as a subscript and a superscript in an expression. The α-Riemann-Christoffel curvature tensor has elements

$$\overset{\alpha}{R}_{ijk\ell}(\boldsymbol{\eta}) = (\partial_i \overset{\alpha}{\Gamma}{}^s_{jk} - \partial_j \overset{\alpha}{\Gamma}{}^s_{ik}) g_{s\ell} + (\overset{\alpha}{\Gamma}_{is\ell} \overset{\alpha}{\Gamma}{}^s_{jk} - \overset{\alpha}{\Gamma}_{js\ell} \overset{\alpha}{\Gamma}{}^s_{ik}), \ (i,j,k,\ell = 1,...,r),$$

where $\overset{\alpha}{\Gamma}{}^s_{jk} = \overset{\alpha}{\Gamma}_{jkt} g^{st}$, g^{st} denotes the $(s,t)^{th}$ element of the inverse of the expected Fisher information matrix and ∂_i denotes $\frac{\partial}{\partial \eta_i}$. The scalar curvature of $\mathcal{S}$ is defined by

$$\overset{\alpha}{C} = \overset{\alpha}{R}_{ijk\ell} g^{i\ell} g^{jk},$$

and the manifold $\mathcal{S}$ is said to be α-flat if $\overset{\alpha}{R}_{ijk\ell}(\boldsymbol{\eta}) = 0$ for all $\boldsymbol{\eta}$, $i,j,k,\ell = 1,...,r$ or equivalently if $\overset{\alpha}{C} = 0$ for all $\boldsymbol{\eta}$. For the submanifold $\mathcal{M}$ of dimension m, and with coordinate system $\boldsymbol{\beta}$, the Riemannian metric tensor, α-connections, the α-Riemann-Christoffel curvature tensor and the scalar curvature are defined analogous to the above equations by replacing i,j,k,ℓ with

a, b, c, d which subscript the components of the parameter vector $\boldsymbol{\beta}$. Further, Amari (1985) has discussed the α-imbedding curvature of $\mathcal{M}$ in $\mathcal{S}$ of which the scalar curvature γ^2 (Efron, 1975) is a special case.

3. COMPUTATIONS FOR $ARFIMA$ PROCESSES

This section describes the derivation of various geometrical quantities for the $ARFIMA(p, d, q)$ process, which is an $(n, m)-$curved exponential family. The computations involve the evaluation of expectations of products of certain derivatives of the logarithm of the exact likelihood in (3). For this, we apply a result from Dahlhaus (1989, Theorem 5.1) that uses an asymptotic result relating Toeplitz matrices to the corresponding spectral density functions of strongly dependent processes. First, we express the $ARFIMA(p, d, q)$ spectral density $s_{\boldsymbol{\beta}}(w)$ in terms of an alternate parametrization. Let ρ_j, $j = 1, ..., p$ and δ_ℓ, $\ell = 1, ..., q$ denote the roots of $\phi(z) = 0$ and $\theta(z) = 0$ respectively. Equation (4) is expressed as

$$s_{\boldsymbol{\tau}}(w) = \frac{\sigma^2}{2\pi} \frac{\{\prod_{j=1}^{q} (1 - \delta_j e^{\mathrm{i}w})(1 - \delta_j e^{-\mathrm{i}w})\}}{\{\prod_{j=1}^{p} (1 - \rho_j e^{\mathrm{i}w})(1 - \rho_j e^{-\mathrm{i}w})\}} \{(1 - e^{\mathrm{i}w})(1 - e^{-\mathrm{i}w})\}^{-d} \qquad (9)$$

where $0 < w \le 2\pi$, $\boldsymbol{\tau} = (\rho_1, ..., \rho_p,\ \delta_1, ..., \delta_q, \sigma^2, d)$, $\mid \rho_j \mid < 1,\ j = 1, ..., p$ and $\mid \delta_\ell \mid < 1,\ \ell = 1, ..., q$. Let $s_{\boldsymbol{\tau}}^{(a)}(w) \equiv \frac{\partial}{\partial \tau_a} s_{\boldsymbol{\tau}}(w)$ and $s_{\boldsymbol{\tau}}^{(a,b)}(w) \equiv \frac{\partial^2}{\partial \tau_a \partial \tau_b} s_{\boldsymbol{\tau}}(w)$, $a, b = 1, ..., m$. The following steps yield the required quantities.

Step 1: In terms of $\boldsymbol{\tau}$,

$$E(\frac{\partial \ell}{\partial \tau_a} \frac{\partial \ell}{\partial \tau_b}) = \frac{1}{2} tr(\Sigma^{-1}\Sigma^a\Sigma^{-1}\Sigma^b),$$

$$E(\frac{\partial \ell}{\partial \tau_a} \frac{\partial \ell}{\partial \tau_b} \frac{\partial \ell}{\partial \tau_c}) = tr(\Sigma^{-1}\Sigma^a\Sigma^{-1}\Sigma^b\Sigma^{-1}\Sigma^c)$$

and

$$E(\frac{\partial^2 \ell}{\partial \tau_a \partial \tau_b} \frac{\partial \ell}{\partial \tau_c}) = \frac{1}{2} tr(\Sigma^{-1}\Sigma^{(a,b)}\Sigma^{-1}\Sigma^c) - tr(\Sigma^{-1}\Sigma^a\Sigma^{-1}\Sigma^b\Sigma^{-1}\Sigma^c)$$

where

$$\Sigma^a = \frac{\partial \Sigma}{\partial \tau_a},\ \Sigma^{(a,b)} = \frac{\partial^2 \Sigma}{\partial \tau_a \partial \tau_b},\ (a, b = 1, ..., m).$$

Step 2: Express the traces in Step 1, up to $o(1)$ as integrals of products of $s_{\boldsymbol{\tau}}(w)$ and suitable derivatives of $s_{\boldsymbol{\tau}}(w)$, using Theorem 5.1, Dahlhaus (1989). Specifically, for the expressions in Step 1, this gives

$$n^{-1} tr(\Sigma^{-1}\Sigma^a\Sigma^{-1}\Sigma^b) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s_{\boldsymbol{\tau}}^{(a)}(w)}{s_{\boldsymbol{\tau}}(w)} \frac{s_{\boldsymbol{\tau}}^{(b)}(w)}{s_{\boldsymbol{\tau}}(w)} dw + o(1),$$

$$n^{-1} tr(\Sigma^{-1}\Sigma^a\Sigma^{-1}\Sigma^b\Sigma^{-1}\Sigma^c) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s_{\boldsymbol{\tau}}^{(a)}(w)}{s_{\boldsymbol{\tau}}(w)} \frac{s_{\boldsymbol{\tau}}^{(b)}(w)}{s_{\boldsymbol{\tau}}(w)} \frac{s_{\boldsymbol{\tau}}^{(c)}(w)}{s_{\boldsymbol{\tau}}(w)} dw + o(1)$$

and

$$n^{-1} tr(\Sigma^{-1}\Sigma^{(a,b)}\Sigma^{-1}\Sigma^c) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s_{\boldsymbol{\tau}}^{(a,b)}(w)}{s_{\boldsymbol{\tau}}(w)} \frac{s_{\boldsymbol{\tau}}^{(c)}(w)}{s_{\boldsymbol{\tau}}(w)} dw + o(1).$$

Note that for the weakly dependent $ARMA$ processes, Taniguchi (1983) provided similar expressions upto $O(n^{-1})$.

Step 3: Evaluate the integrals obtained in Step 2. Terms that do not involve d lead to integrals with rational integrands which may be evaluated using Hille's (1959) residue theorem. When derivatives of $s_{\boldsymbol{\tau}}(w)$ with respect to d are evaluated, the integrands are not rational functions, and we use special complex integration techniques, although for certain simple terms, formulas may be obtained from Gradshteyn and Ryzhik (1965). In general, numerical integration may be used if *analytical expressions* are not necessary for the intended application of these quantities.

Step 4: Step 3 yields analytic expressions for the geometric quantities in terms of $\boldsymbol{\tau}$. Based on the relationship

$$(1 - \phi_1 z - \cdots - \phi_p z^p) = (1 - \rho_1 z)(1 - \rho_2 z) \cdots (1 - \rho_p z),$$

we obtain the elements of the Jacobian matrix of the transformation from $(\rho_1, ..., \rho_p)$ to $(\phi_1, ..., \phi_p)$ as

$$\frac{\partial \rho_j}{\partial \phi_k} = \frac{-\rho_j^{p-k}}{\prod\limits_{\substack{\ell=1 \\ \ell \neq j}}^{p} (\rho_\ell - \rho_j)}. \tag{10}$$

A similar result holds for the transformation from $(\delta_1, ..., \delta_q)$ to $(\theta_1, ..., \theta_q)$.

Explicit expressions for the elements of the Riemannian metric tensor $g_{ij}(\boldsymbol{\tau})$ and $\alpha-$connections $\overset{\alpha}{\Gamma}_{ijk}(\boldsymbol{\tau})$ in (5) and (6) that are computed using these expectations are shown in the Appendix. The geometric quantities in terms of the coordinate system $\boldsymbol{\beta}$ are then computed using (10) and the coordinate transformation laws given in (7) and (8). Although for simplicity, we have focused in this article on real roots of $\phi(z) = 0$ and $\theta(z) = 0$, this is not binding and the procedure holds for complex roots as well (see Ravishanker et al., 1990 for the $ARMA$ case). Table 1 shows the non-zero elements of $g_{ij}(\boldsymbol{\beta})$ and $\overset{\alpha}{\Gamma}_{ijk}(\boldsymbol{\beta})$ for the fractional Gaussian noise process with $\boldsymbol{\beta} = (d, \sigma^2)$.

TABLE I. Geometrical Quantities for the $ARFIMA(0, d, 0)$ Process

Riemmanian Metric Tensor	$\alpha-$Connections
$g_{11} = \pi^2/6$	$\overset{\alpha}{\Gamma}_{111} = -6\alpha \sum_{k=1}^{\infty} 1/k^3$
$g_{22} = 1/2\sigma^4$	$\overset{\alpha}{\Gamma}_{112} = -\alpha\pi^2/6\sigma^2$
	$\overset{\alpha}{\Gamma}_{121} = -(1+\alpha)\pi^2/6\sigma^2$
	$\overset{\alpha}{\Gamma}_{122} = -(1+\alpha)/2\sigma^8$
$\overset{\alpha}{C} = (4/\sigma^4 + 2/\sigma^2) + \alpha(8/\sigma^4 + 6/\sigma^2 + 6)$ $+\alpha^2[4/\sigma^4 + 4/\sigma^2 + 8 + \frac{15552}{\pi^6}(\sum_{k=1}^{\infty} 1/k^3)^2]$	

The result that $g_{11} = \pi^2/6$ was noted previously by Kashyap and Eom (1988). If σ^2 is assumed to be known (and equal to 1, say), then $\boldsymbol{\beta} = d$, and the statistical curvature (Efron, 1975) is given by

$$\gamma_d^2 = \frac{2\frac{\pi^2}{3}Q_6^* - 36(\sum_{k=1}^{\infty} 1/k^3)^2}{n\pi^6/216}, \tag{11}$$

where $Q_6^* = [-\frac{17}{120}\pi^4 + 30 \sum_{k=1}^{\infty} 1/k^4 - 12 \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} 1/\ell(k+\ell)^3]$, so that $\gamma_d^2 \simeq \frac{11.1249}{n}$. Since a one-dimensional manifold is always $\alpha-$ flat for any α, the $\alpha-$affine parameter $\overset{\alpha^*}{d}$ is determined from d by solving a differential equation (Amari, 1985, equation (5.20)) as

$$\overset{\alpha^*}{d} = \begin{cases} \frac{-\pi^2}{36\alpha(\sum_{k=1}^{\infty} 1/k^3)} \exp(-\frac{36\alpha}{\pi^2} \sum_{k=1}^{\infty} (1/k^3)d\} & \alpha \neq 0 \\ d & \alpha = 0. \end{cases} \tag{12}$$

4. APPLICATION TO ASYMPTOTIC INFERENCE

The geometrical quantities computed in Section 3 enable us to study higher-order asymptotic inference for $ARFIMA(p,d,q)$ processes. In this section, preliminary results that illustrate the role of geometry in inference are presented for the $ARFIMA(0,d,0)$ process (see equation (2)), with $\beta_1 = d$, and $\beta_2 = \sigma^2$. In particular, some results are shown relating to the asymptotic bias in the MLEs of the parameters, the Bartlett correction factor to the likelihood ratio test statistic, and second-order asymptotic efficiency of the estimates. Clearly, more work is needed in this direction.

Dahlhaus (1989) proved that the exact MLE $\widehat{\boldsymbol{\beta}}_n = (\widehat{\beta}_1, \cdots, \widehat{\beta}_m)$ is consistent, first-order efficient, and has an asymptotic normal distribution with mean $\boldsymbol{\beta}$. Based on the quantity (see Amari, 1985, p.146)

$$B_a = -\frac{1}{2n} \sum_c \sum_d \sum_e g^{ae} g^{cd} \overset{-1}{\Gamma}_{cde}, \qquad a = 1, \cdots, m$$

where g^{ij} denotes the (i,j)th element of the inverse of the expected Fisher information matrix, we compute

$$B_1 = \frac{-108(\sum_{k=1}^{\infty} 1/k^3)}{n\pi^4} \simeq -1.33/n$$

and

$$B_2 \simeq -\sigma^2/n.$$

B_a may be interpreted as the leading term in the asymptotic bias in $\widehat{\beta}_a$ and is common to all first-order efficient estimators of β_a. The quantity B_1 is constant and negative, suggesting that d tends to be underestimated. Our results compare in magnitude with the simulation results obtained by Cheung and Newbold (1994) on the bias in the MLE of d from the $ARFIMA(0,d,0)$ process for time-domain maximum likelihood with the true mean removed, although they report an inverse relation between the bias and the d value.

The likelihood ratio statistic LR for testing $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$ is given by

$$LR = 2\{\ell(\widehat{\boldsymbol{\beta}}, Z_n) - \ell(\boldsymbol{\beta}_0, Z_n)\}$$

Under H_0, LR has an asymptotic χ^2 distribution with m_0 degrees of freedom where m_0 is the number of restricted parameters (Dahlhaus, 1989). It is well known that the rate of convergence to the χ^2 distribution may be improved by using a scaled likelihood ratio test statistic LR^*:

$$LR^* = \frac{LR}{1+R/m_0}$$

where an explicit expression for the *Bartlett correction factor* R is given in Barndorff-Nielsen and Blaesild (1986). For testing $H_0 : d = 0$, σ^2 unrestricted in the $ARFIMA(0,d,0)$ model, we compute R based on the geometrical expressions in Section 3 as

$$R = \frac{-36}{\pi^6}[\pi^2 Q_3^* - 90(\sum_{k=1}^{\infty} 1/k^3)^2].$$

Extension of these results to the general $ARFIMA$ processes, and study of the rates of convergence are topics for future research. It will also be useful to study properties of second-order asympototically efficient estimates of $ARFIMA$ model parameters based on the geometrical quantities (see Akahira and Takeuchi, 1981 and Taniguchi, 1991, for results pertaining to $ARMA$ processes).

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APPENDIX

For the $ARFIMA(p, d, q)$ process, the non-zero elements of the Riemannian metric tensor (5) under the parametrization $\boldsymbol{\tau} = (\rho_1, ..., \rho_p, \delta_1, ..., \delta_q, \sigma^2, d)$ are

$$g_{\rho_j,\rho_\ell} = \frac{1}{(1-\rho_j\rho_\ell)}, (j, \ell = 1, ..., p)$$

$$g_{\delta_j,\delta_\ell} = \frac{1}{(1-\delta_j\delta_\ell)}, (j, \ell = 1, ..., q)$$

$$g_{\rho_j,\delta_\ell} = \frac{-1}{(1-\rho_j\delta_\ell)}, (j = 1, ..., p,\ \ \ell = 1, ..., q)$$

$$g_{d,d} = \frac{\pi^2}{6}$$

$$g_{\rho_j,d} = -\frac{1}{\rho_j}\ln(1-\rho_j), (j = 1, ..., p)$$

$$g_{\delta_\ell,d} = \frac{1}{\delta_\ell}\ln(1-\delta_\ell), (\ell = 1, ..., q)$$

$$g_{\sigma^2,\sigma^2} = \frac{1}{2\sigma^4},$$

where $ln(x)$ denotes the natural logarithm of x. In the same parametrization, the non-zero elements of the α-connections in (6) are:

$$\underset{d,d,d}{\overset{\alpha}{\Gamma}} = -6\alpha \sum_{k=1}^{\infty} \frac{1}{k^3}; \qquad \underset{d,d,\sigma^2}{\overset{\alpha}{\Gamma}} = \frac{-\alpha\pi^2}{6\sigma^2};$$

$$\underset{d,\rho_j,d}{\overset{\alpha}{\Gamma}} = \underset{d,d,\rho_j}{\overset{\alpha}{\Gamma}} = \alpha Q_1^*(\rho_j), (j = 1, ..., p);$$

$$\underset{d,\delta_j,d}{\overset{\alpha}{\Gamma}} = \underset{d,d,\delta_j}{\overset{\alpha}{\Gamma}} = -\alpha Q_1^*(\delta_j), (j = 1, ..., q); \qquad \underset{d,\sigma^2,d}{\overset{\alpha}{\Gamma}} = \frac{-(1+\alpha)\pi^2}{6\sigma^2};$$

$$\underset{d,\sigma^2,\rho_j}{\overset{\alpha}{\Gamma}} = \frac{(\alpha+1)\ln(1-\rho_j)}{\sigma^2\rho_j},\ (j = 1, ..., p);$$

$$\underset{d,\sigma^2,\delta_j}{\overset{\alpha}{\Gamma}} = \frac{-(\alpha+1)\ln(1-\delta_j)}{\sigma^2\delta_j},\ (j = 1, ..., q);$$

$$\underset{d,\rho_j,\sigma^2}{\overset{\alpha}{\Gamma}} = \underset{\sigma^2,\rho_j,d}{\overset{\alpha}{\Gamma}} = \frac{\alpha\ln(1-\rho_j)}{\sigma^2\rho_j},\ (j = 1, ..., p);$$

$$\underset{d,\rho_j,\rho_k}{\overset{\alpha}{\Gamma}} = \begin{cases} \alpha Q_2^*(\rho_j),\ j=k \\ \alpha Q_3^*(\rho_j,\rho_k),\ j\neq k,\ j,k=1,...,p; \end{cases}$$

$$\underset{d,\rho_j,\delta_k}{\overset{\alpha}{\Gamma}} = \underset{\rho_j,\delta_k,d}{\overset{\alpha}{\Gamma}} = -\alpha Q_3^*(\rho_j,\delta_k),\ (j = 1, ..., p, k = 1, ..., q);$$

$$\underset{d,\delta_j,\sigma^2}{\overset{\alpha}{\Gamma}} = \underset{\sigma^2,\delta_j,d}{\overset{\alpha}{\Gamma}} = \frac{-\alpha\ln(1-\delta_j)}{\sigma^2\delta_j},\ (j=1,...,q);$$

$$\underset{d,\delta_j,\delta_k}{\overset{\alpha}{\Gamma}} = \begin{cases}\alpha Q_2^*(\delta_j),\ j=k\\ \alpha Q_3^*(\delta_j,\delta_k),\ j\neq k,\ j,k=1,...,q;\end{cases}$$

$$\underset{d,\delta_j,\rho_k}{\overset{\alpha}{\Gamma}} = -\alpha Q_3^*(\delta_j,\rho_k),\ (k=1,...,p,j=1,...,q);$$

$$\underset{\sigma^2,\sigma^2,\sigma^2}{\overset{\alpha}{\Gamma}} = \frac{-(1+\alpha)}{2\sigma^6}; \qquad \underset{\sigma^2,\rho_j,\rho_k}{\overset{\alpha}{\Gamma}} = \underset{\rho_j,\rho_k,\sigma^2}{\overset{\alpha}{\Gamma}} = \frac{-\alpha}{\sigma^2(1-\rho_j\rho_k)},\ (j,k=1,...,p);$$

$$\underset{\sigma^2,\rho_j,\delta_k}{\overset{\alpha}{\Gamma}} = \underset{\rho_j,\delta_k,\sigma^2}{\overset{\alpha}{\Gamma}} = \underset{\sigma^2,\delta_k,\rho_j}{\overset{\alpha}{\Gamma}} = \frac{\alpha}{\sigma^2(1-\rho_j\delta_k)},\ (j=1,...,p,k=1,...,q);$$

$$\underset{\sigma^2,\delta_j,\delta_k}{\overset{\alpha}{\Gamma}} = \underset{\delta_j,\delta_k,\sigma^2}{\overset{\alpha}{\Gamma}} = \frac{-\alpha}{\sigma^2(1-\delta_j\delta_k)},\ (j,k=1,...,q);$$

$$\underset{\rho_j,\rho_k,d}{\overset{\alpha}{\Gamma}} = \begin{cases}Q_4^*(\rho_j),\ j=k\\ \alpha Q_3^*(\rho_j,\rho_k),\ j\neq k,\ j,k=1,...,p;\end{cases}$$

$$\underset{\rho_j,\rho_k,\rho_\ell}{\overset{\alpha}{\Gamma}} = \begin{cases}\frac{-\alpha(\rho_j+\rho_k+\rho_\ell-3\rho_j\rho_k\rho_\ell)}{(1-\rho_j\rho_k)(1-\rho_j\rho_\ell)(1-\rho_k\rho_\ell)},\ k\neq j\\ \frac{-2\alpha\rho_j+(1-\alpha)\rho_\ell+(3\alpha-1)\rho_j^2\rho_\ell}{(1-\rho_j\rho_\ell)^2(1-\rho_j^2)},\ k=j;\ j,k,\ell=1,...,p;\end{cases}$$

$$\underset{\rho_k,\rho_\ell,\delta_j}{\overset{\alpha}{\Gamma}} = \begin{cases}\frac{\alpha(\rho_k+\rho_\ell+\delta_j-3\rho_k\rho_\ell\delta_j)}{(1-\delta_j\rho_k)(1-\delta_j\rho_\ell)(1-\rho_{k\rho\ell})},\ k\neq\ell\\ \frac{2\alpha\rho_k-(1-\alpha)\delta_j+(1-3\alpha)\rho_k^2\delta_j}{(1-\rho_k^2)(1-\rho_k\delta_j)^2},\ k=\ell\end{cases} \quad (k,\ell=1,...,p,\ j=1,...,q);$$

$$\underset{\delta_\ell,\rho_j,\rho_k}{\overset{\alpha}{\Gamma}} = \left\{\frac{\alpha(\rho_j+\rho_k+\delta_\ell-3\rho_{j\rho k}\delta_\ell)}{(1-\rho_j\rho_k)(1-\rho_j\delta_\ell)(1-\rho_k\delta_\ell)},\quad j,k=1,...,p,\ \ell=1,...,q;\right.$$

$$\underset{\rho_\ell,\delta_j,\delta_k}{\overset{\alpha}{\Gamma}} = \frac{-\alpha(\rho_\ell+\delta_j+\delta_k-3\rho_\ell\delta_j\delta_k)}{(1-\rho_\ell\delta_j)(1-\rho_\ell\delta_k)(1-\delta_j\delta_k)},\ \ell=1,...,p,\ j,k=1,...,q;$$

$$\underset{\delta_j,\delta_k,d}{\overset{\alpha}{\Gamma}} = \begin{cases}Q_5^*(\delta_j),\ j=k\\ \alpha Q_3^*(\delta_j,\delta_k),\ j\neq k,\ j,k=1,...,q;\end{cases}$$

$$\underset{\delta_k,\delta_\ell,\rho_j}{\overset{\alpha}{\Gamma}} = \begin{cases}\frac{-\alpha(\rho_j+\delta_k+\delta_\ell-3\rho_j\delta_k\delta_\ell)}{(1-\rho_j\delta_k)(1-\rho_j\delta_\ell)(1-\delta_k\delta_\ell)},\ k\neq\ell\\ \frac{-(1+\alpha)\rho_j-2\alpha\delta_\ell+(1+3\alpha)\rho_j\delta_\ell^2}{(1-\rho_j\delta_\ell)^2(1-\delta_\ell^2)},\ k=\ell;\ j=1,...,p,k,\ell=1,...,q;\end{cases}$$

$$\overset{\alpha}{\Gamma}_{\delta_j\delta_k,\delta_\ell} = \begin{cases} \frac{\alpha(\delta_j+\delta_k+\delta_\ell-3\delta_j\delta_k\delta_\ell)}{(1-\delta_j\delta_k)(1-\delta_j\delta_\ell)(1-\delta_k\delta_\ell)}, & k\neq j \\ \frac{(1+\alpha)\delta_\ell+2\alpha\delta_j-(1+3\alpha)\delta_j^2\delta_\ell}{(1-\delta_j\delta_\ell)^2(1-\delta_j^2)}, & k=j;\ j,k,\ell=1,\ldots,q; \end{cases}$$

where $Q_1^*(\gamma_j) = \{c_1(c_2^2-\pi^2) - c_1(\ln 2)^2 + 4c_1(\pi^2/48+S_1)\ -2c_1(c_3-c_4+S_2) - 2(S_3+2S_4)/(1+\gamma_j^2)\}$,

$$Q_2^*(\gamma_j) = \frac{1}{\gamma_j(\gamma_j-1)} + \frac{(3\gamma_j^6+5\gamma_j^4+\gamma_j^2-1)}{\gamma_j^2(1-\gamma_j^2)(1+\gamma_j^2)^2}\ln(1-\gamma_j),$$

$$Q_3^*(\gamma_j,\gamma_k) = \frac{(1+\gamma_j^2-2\gamma_j\gamma_k)}{(1-\gamma_j\gamma_k)(\gamma_j-\gamma_k)\gamma_j}\ln(1-\gamma_j) + \frac{(1+\gamma_k^2-2\gamma_j\gamma_k)}{(1-\gamma_j\gamma_k)(\gamma_k-\gamma_j)\gamma_k}\ln(1-\gamma_k),$$

$$Q_4^*(\rho_j) = \frac{2[(1+4\rho_j^2+5\rho_j^4-2\rho_j^6)\ln(1-\rho_j)+\rho_j(1+\rho_j-3\rho_j^2-2\rho_j^3+\rho_j^4+\rho_j^5)]}{\rho_j^2(1-\rho_j^2)^3} + (1+\alpha)Q_2^*(\rho_j),$$

$$Q_5^*(\delta_j) = \frac{-2\ln(1-\delta_j)}{(1-\delta_j^2)} - (1+\alpha)Q_2^*(\delta_j),$$

$$c_1 = \gamma_j/(1+\gamma_j^2); \qquad c_2 = \ln((1-\gamma_j)^2/\gamma_j),$$

$$c_3 = 2\ln(1-\gamma_j)\ln((1-\gamma_j)/\gamma_j); \qquad c_4 = \ln\gamma_j\ln((1-\gamma_j)/\gamma_j);$$

$$S_1 = \sum_{k=1}^{\infty}\sum_{\ell=1}^{2k}\frac{(-1)^{k+\ell+1}}{k\ell}\binom{2k}{\ell} \simeq -0.8565;$$

$$S_2 = \sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\frac{\binom{k}{\ell}}{k\ell}\{(-1)^\ell[(1-\gamma_j)^k+2\gamma_j^k] - [\gamma_j^\ell(1-\gamma_j)^{k-\ell}+2\gamma_j^{k-\ell}(1-\gamma_j)^\ell]\};$$

$$S_3 = \sum_{k=1}^{\infty}\gamma_j^{k-1}/k^2; \qquad S_4 = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{k-\ell+1}\gamma_j^k}{(k-\ell+1)^2}; \qquad S_5 = \sum_{k=0}^{\infty}\gamma_j^k/(k+1)^2.$$

The quantities $Q_i^*(\rho_j)$ and $Q_i^*(\delta_j)$ are obtained by substituting ρ_j and δ_j respectively for γ_j in the formulas for $Q_i^*(\gamma_j)$ and their components, $i = 1, 2$, while $Q_3^*(\rho_j, \rho_k)$, $Q_3^*(\delta_j, \delta_k)$ and $Q_3^*(\delta_j, \rho_k)$ are obtained likewise by suitable substitutions in $Q_3^*(\gamma_j, \gamma_k)$. Note that we have retained the parameter symbols as subscripts on the Riemannian metric tensor and α-connections for ease of reading. Correspondence with the notation of Section 2 is achieved by setting $(d, \sigma^2, \rho_1, ..., \rho_p,\ \delta_1, ..., \delta_q) \equiv (1, 2, 3, ..., p+2, p+3, ..., p+q+2)$.