ST235: A Note on Poisson Processes

Definition 1 of a PP: A stochastic process \( \{N(t), t \geq 0\} \) is a PP with rate \( \lambda \) if

(i) \( N(0) = 0 \),

(ii) \( \{N(t)\} \) has independent and stationary increments, and

(iii) for all \( s, t \geq 0 \),

\[
P[N(t + s) - N(s) = n] = \frac{\exp(-\lambda t)(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots;
\]

in particular,

\[
P[N(t) = n] = \frac{\exp(-\lambda t)(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots;
\]

so that \( E(N(t)) = \lambda t \).

Definition 2 of a PP: A stochastic process \( \{N(t), t \geq 0\} \) is a PP with rate \( \lambda \) if

(i') \( N(0) = 0 \),

(ii') \( \{N(t)\} \) has independent and stationary increments,

(iii') \( P[N(h) = 1] = \lambda h + o(h), \) and

(iv') \( P[N(h) \geq 2] = o(h). \)

Definition 3 of a PP: Consider a sequence \( \{T_n, n \geq 1\} \) of iid \( \exp(\lambda) \) random variables. Define a counting process by requiring that the \( n \)th event of this process occurs at time \( S_n = T_1 + \cdots + T_n \). Let \( N(t) = \max\{n : S_n \leq t\} \). Then, \( \{N(t), t \geq 0\} \) is a PP with rate \( \lambda \).

Result: Definition 1 and Definition 2 of a PP are equivalent.

To see that Definition 2 implies Definition 1, let \( P_n(t) = P(N(t) = n) \), i.e., the probability of \( n \) events occurring by time \( t \). Now, \( P(0 \text{ events by time } t + h) \) is equal to the probability of 0 events up to time \( t \) and 0 events in the interval \( (t, t+h) \), i.e.,

\[
P_0(t + h) = P(N(t + h) = 0) = P(N(t) = 0, N(t + h) - N(t) = 0) = P(N(t) = 0)P(N(t + h) - N(t) = 0) = P_0(t)[1 - \lambda h + o(h)]
\]
by (ii') of Definition 2. Hence,

\[
\frac{P_0(t + h) - P_0(t)}{h} = -\lambda P_0(t) + o(h),
\]

\[
\lim_{h \to 0} \frac{P_0(t + h) - P_0(t)}{h} = \frac{d}{dt} P_0(t) = P'_0(t) = -\lambda P_0(t), \text{i.e.,}
\]

\[
\frac{P'_0(t)}{P_0(t)} = -\lambda, \text{i.e.,}
\]

\[
\int \frac{P'_0(t)}{P_0(t)} dt = -\lambda \int dt, \text{i.e.,}
\]

\[
\log P_0(t) = -\lambda t + c,
\]

where \(c\) is a constant of integration. We can write

\[
P_0(t) = K \exp(-\lambda t),
\]

where \(K\) is a constant. From (i') of Definition 2, \(P_0(0) = P(N(0) = 0) = 1\), so that \(P_0(0) = K \exp(-0) = 1\), so that \(K = 1\), and

\[
P_0(t) = P(N(t) = 0) = \exp(-\lambda t).
\]

Similarly, for \(n > 0\),

\[
P_n(t + h) = P(N(t + h) = n)
\]

\[
= P(N(t) = n, N(t + h) - N(t) = 0) + P(N(t) = n - 1, N(t + h) - N(t) = 1)
\]

\[
+ \sum_{k=2}^{n} P(N(t) = n - k, N(t + h) - N(t) = k)
\]

\[
= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h), \text{(by (ii') and (iv'))}
\]

\[
= P_n(t)(1 - \lambda h) + \lambda h P_{n-1}(t) + o(h) \quad \text{(by (iii') and (iv'))}.
\]

Hence,

\[
\frac{P_n(t + h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + o(h)
\]

\[
P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \text{i.e.,}
\]

\[
\exp(\lambda t)[P'_n(t) + \lambda P_n(t)] = \lambda \exp(\lambda t) P_{n-1}(t), \text{i.e.,}
\]

\[
\frac{d}{dt}[\exp(\lambda t) P_n(t)] = \lambda \exp(\lambda t) P_{n-1}(t).
\]

When \(n = 1\), we have

\[
\frac{d}{dt}[\exp(\lambda t) P_1(t)] = \lambda \exp(\lambda t) P_0(t)
\]

\[
= \lambda \exp(\lambda t) \exp(-\lambda t) = \lambda.
\]

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Or, integrating,

\[ P_1(t) \exp(\lambda t) = \lambda t + c \text{, i.e.,} \]

\[ P_1(t) = \exp(-\lambda t)(\lambda t + c). \]

Now, \( P_1(0) = 0 = \exp(0)(0 + c) \), so that \( c = 0 \). Hence,

\[ P_1(t) = \lambda t \exp(-\lambda t). \]

Next, suppose

\[ P_{n-1}(t) = \frac{\exp(-\lambda t)(\lambda t)^{n-1}}{(n-1)!}. \]

We show this holds for \( n \) by mathematical induction. We see that

\[
\frac{d}{dt}[\exp(\lambda t)P_n(t)] = \lambda \exp(\lambda t)P_{n-1}(t)
\]

\[ = \lambda \exp(\lambda t)\frac{\exp(-\lambda t)(\lambda t)^{n-1}}{(n-1)!}
\]

\[ = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}, \]

or by integration,

\[ \exp(\lambda t)P_n(t) = \frac{\lambda^n}{(n-1)!} \int t^{n-1} dt
\]

\[ = \frac{\lambda^n}{(n-1)!} \frac{t^n}{n} + c
\]

\[ = \frac{(\lambda t)^n}{n!} + c. \]

Since \( P_n(0) = 0 \), we get \( c = 0 \), so that

\[ P_n(t) = \frac{\exp(-\lambda t)(\lambda t)^n}{n!}, \]

i.e., we have shown that Definition 2 implies Definition 1.

In Exercise 5.27, you are asked to show that Definition 1 implies Definition 2. The following steps may be used. Using (ii) and (i) of Definition 1, show that

\[ P[N(h) = 0] = \exp(-\lambda h), P[N(h) - N(0) = 1] = P(N(h) = 1) = \lambda h \exp(-\lambda h), \]

and \( P(N(h) \geq 2) = 1 - \exp(-\lambda h) - \lambda h \exp(-\lambda h) \). The results follow by taking \( \lim_{h \to 0} \).

Alternate Derivations of the distribution of \( S_n \).

**Derivation 1:** Note that \( P(S_n \leq t) \) is \( P(\text{Number of events that occur by time } t \text{ is at least } n) \), so that \( S_n \leq t \iff N(t) \geq n \). Hence,

\[ F_{S_n}(t) = P(S_n \leq t) = P(N(t) \geq n) = \sum_{j=n}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^j}{j!}, \]
\[
fsn(t) = \frac{d}{dt} F_{Sn}(t) = \frac{d}{dt} \sum_{j=0}^{\infty} \frac{(-\lambda t)^j}{j!} = \lambda \exp(-\lambda t) \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t > 0,
\]

which is a Gamma\((n, \lambda)\) pdf.

**Derivation 2:** We can express \(P(n\text{th} \text{ arrival occurs in the time interval } (t, t + h))\) as

\[
P(t < S_n < t + h) = P[N(t) = n - 1, 1 \text{ event in } (t, t + h)] + o(h) = P[N(t) = n - 1]P[1 \text{ event in } (t, t + h)] + o(h) = \frac{\exp(-\lambda t)(\lambda t)^{n-1}}{(n-1)!}[\lambda h + o(h)] + o(h) = \lambda \exp(-\lambda t) \frac{(\lambda t)^{n-1}}{(n-1)!} h + o(h).
\]

Hence,

\[
fsn(t) = \lim_{h \to 0} \frac{P(t < S_n < t + h)}{h} = \lambda \exp(-\lambda t) \frac{(\lambda t)^{n-1}}{(n-1)!},
\]

which is the pdf of a Gamma\((n, \lambda)\) variable.

**Conditional Distribution of Arrival Times under a HPP:** Given \(N(t) = n\), the \(n\) arrival times \(S_1, \ldots, S_n\) have the same distribution as the order statistics corresponding to \(n\) independent random variables from a Uniform\((0, t)\) distribution.

To show this, let \(\{N(t), t \geq 0\}\) be a PP with rate \(\lambda\). For \(s \leq t\),

\[
P(T_1 < s|N(t) = 1) = \frac{P(1 \text{ event in } (0, s), 0 \text{ events in } (s, t))}{P[N(t) = 1]} = \frac{\exp(-\lambda s)(\lambda s)!}{\exp(-\lambda t)\lambda t} = \frac{s}{t},
\]

which is the cdf of a Uniform\((0, t)\) random variable. Hence \(S_1 = T_1\) has a Uniform\((0, t)\) distribution, and

\[
fs1(s_1|N(t) = 1) = 1/t, \quad 0 < s_1 < t.
\]

Since \(S_n = \sum_{i=1}^{n} T_i\), we have \(T_1 = S_1; T_2 = S_2 - S_1; T_3 = S_3 - S_2; \ldots; T_n = S_n - S_{n-1}\). This transformation has Jacobian \(|J| = 1\). Now, \(\{S_1 = s_1, S_2 = s_2, \ldots, S_n = s_n, N(t) = n\}\) is equivalent to \(\{T_1 = s_1, T_2 = s_2 - s_1, \ldots, T_n = \}

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\[ s_n - s_{n-1}, T_{n+1} > t - s_n \]. Hence,

\[
\begin{align*}
 f_{S_1, \ldots, S_n}(s_1, \ldots, s_n | N(t) = n) &= \frac{f(s_1, \ldots, s_n, n)}{P(N(t) = n)} \\
 &= \frac{f_{T_1, \ldots, T_{n+1}}(s_1, s_2 - s_1, \ldots, T_{n+1} > t - s_n)}{\exp(-\lambda t)(\lambda t)^n / n!} \\
 &= \frac{f_{T_1}(s_1)f_{T_2}(s_2 - s_1) \cdots f_{T_n}(s_n - s_{n-1})(1 - F_{T_{n+1}}(t - s_n))}{\exp(-\lambda t)(\lambda t)^n / n!} \\
 &= \frac{\lambda \exp(-\lambda s_1) \cdots \lambda \exp(-\lambda(s_n - s_{n-1}) \exp(-\lambda(t - s_n))}{\exp(-\lambda t)(\lambda t)^n / n!} \\
 &= \frac{n!}{t^n} \quad 0 < s_1 < s_2 < \cdots < s_n < t.
\]

This is the joint pdf of \( n \) order statistics from a Uniform(0, t) distribution.

**Distribution of \( S_n \) under a NHPP:** Let \( S_n \) denote the waiting time until the \( n \)th event of a NHPP with intensity function \( \lambda(t) \). Then,

\[
f_{S_n}(t) = \lambda(t) \exp(-m(t)) \left[ \frac{m(t)^{n-1}}{(n-1)!} \right]
\]

where \( m(t) = \int_0^t \lambda(u)du \) is the mean value function. As we did with the HPP, we give two derivations of this distribution.

**Derivation 1:** Note that \( S_n \leq t \) iff \( N(t) \geq n \). Hence,

\[
F_{S_n}(t) = P(S_n \leq t) = P(N(t) \geq n) = \sum_{j=n}^{\infty} \frac{\exp(-m(t))(m(t))^j}{j!},
\]

and differentiating,

\[
f_{S_n}(t) = \sum_{j=n}^{\infty} \frac{1}{j!} \frac{d}{dt} \exp(-m(t))(m(t))^j
\]

\[
= \sum_{j=n}^{\infty} \frac{1}{j!} \{ \exp(-m(t))j[m(t)]^{j-1}m'(t) + [m(t)]^j [- \exp(-m(t)m'(t)]
\]

and \( m'(t) = \frac{d}{dt}m(t) = \frac{d}{dt} \int_0^t \lambda(u)du = \lambda(t) \). Hence,

\[
f_{S_n}(t) = \exp(-m(t))\lambda(t) \left\{ \sum_{j=n}^{\infty} \frac{(m(t))^{j-1}}{(j-1)!} - \frac{[m(t)]^j}{j!} \right\},
\]

from which the required result follows after simplification.

**Derivation 2:** To see this, note that similar to the HPP case,

\[
P(t < S_n < t+h) = P[N(t) = n - 1, 1 \text{ event in } (t, t+h)] + o(h)
\]

\[
= P[N(t) = n - 1]P[1 \text{ event in } (t, t+h)] + o(h)
\]

\[
= \frac{\exp(-m(t))(m(t))^{n-1}}{(n-1)!} \left[ \lambda(t) + o(h) \right] + o(h)
\]

\[
= \lambda(t) \exp(-m(t)) \left[ \frac{(m(t))^{n-1}}{(n-1)!} \right] h + o(h).
\]
Hence,

\[ f_{S_n}(t) = \lim_{h \to 0} \frac{P(t < S_n < t+h)}{h} = \lambda(t) \exp(-m(t)) \frac{(m(t))^{n-1}}{(n-1)!}. \]