1. B&D, 2.11: 10 points

We are given \( n = 100 \) observations from an AR(1) process with unknown mean \( \mu, \phi = 0.6, \sigma^2 = 2 \), and \( \bar{x} = 0.271 \). For an AR(1) process, the ACVF function is for \( h = 0, \pm 1, \cdots \),

\[
\gamma(h) = \sigma^2 \frac{\phi^{|h|}}{1 - \phi^2}
\]

We know that

\[
Var(\bar{X}) = \frac{1}{n} \sum_{h=-\infty}^{\infty} \gamma(h)
\]

where

\[
\sum_{h=-\infty}^{\infty} \gamma(h) = \frac{\sigma^2}{(1 - \phi^2)} \sum_{h=-\infty}^{\infty} \phi^{|h|}
\]

\[
= \frac{\sigma^2}{(1 - \phi^2)} \left[ 1 + 2 \sum_{h=0}^{\infty} \phi^h \right]
\]

\[
= \frac{\sigma^2}{(1 - \phi^2)} \left[ 1 + \frac{2\phi}{1 - \phi} \right]
\]

\[
= \frac{\sigma^2}{(1 - \phi)^2} = \frac{2}{(1 - 0.6)^2}
\]

The 95% C.I. for \( \mu \) is

\[
\bar{x} \pm 1.96 \times \sqrt{\frac{2}{(0.4)(10)}}, \text{i.e.,}
0.271 \pm 1.96 \times \sqrt{\frac{2}{(0.4)(10)}} = (-0.422, 0.964)
\]

2. B&D, 2.12: 10 points

We are given \( n = 100 \) observations from an MA(1) process with unknown mean \( \mu, \theta = -0.6, \sigma^2 = 1 \), and \( \bar{x} = 0.157 \). For an MA(1) process, the ACVF function is for \( h = 0, \pm 1, \cdots \),

\[
\gamma(h) = \begin{cases} 
(1 + \theta^2)\sigma^2 & h = 0 \\
\theta\sigma^2 & |h| = 1 \\
0 & |h| \geq 2 
\end{cases}
\]

Then,
\[ \text{Var}(\bar{X}) = \frac{1}{n} \sum_{h=-\infty}^{\infty} \gamma(h) \]

Then,
\[ \frac{1}{n} \sum_{h=-\infty}^{\infty} \gamma(h) = \frac{\sigma^2}{n} [(1 + \theta^2) + 2\theta] = 0.16 \]

The 95\% C.I. for \( \mu \) is
\[ 0.157 \pm (1.96)(0.4)/10 = (0.0786, 0.2354) \]

3. B&D, 3.1: 25 points

(a) AR(2) process. The roots of \( (1 + 0.2z - 0.48z^2) = 0 \) are \(-1.25\) and \(1.66\), hence the process is causal. Alternatively, for an AR(2), we can check the conditions on the parameters, viz., \( \phi_2 + \phi_1 < 1; \phi_2 - \phi_1 < 1; -1 < \phi_2 < 1 \). These are met when \( \phi_1 = 0.2 \) and \( \phi_2 = -0.48 \). The AR process is always invertible.

(b) ARMA(2,2) process. The roots of the AR polynomial \( (1 + 1.9z + 0.88z^2) = 0 \) are \(-0.909\) and \(-1.25\); only one of them exceeds one in absolute value; hence the process is not causal. In terms of using the conditions on \( \phi_1 \) and \( \phi_2 \), note that \( \phi_2 - \phi_1 = 1.02 > 1 \). The roots of the MA polynomial \( (1 + 0.2z + 0.7z^2) = 0 \) are conjugate complex, equal to \(-0.14 \pm 1.18i\); the process is invertible. The coefficients satisfy the conditions \( \theta_2 + \theta_1 < 1; \theta_2 - \theta_1 < 1; -1 < \theta_2 < 1 \).

(c) ARMA(1,1). The process is not invertible since \( \theta = 1.2 > 1 \). The process is causal since \( \phi = 0.6 < 1 \).

(d) AR(2) Process. The process is invertible. Since the roots of \( 1 + 1.8z + 0.81z^2 = 0 \) are \(-1.111\) (repeated), outside the unit circle, the process is causal. Or check that the coefficients satisfy the conditions \( \phi_2 + \phi_1 < 1; \phi_2 - \phi_1 < 1; -1 < \phi_2 < 1 \).

(c) ARMA(1,2) process. The process is invertible since the roots of the MA polynomial \( 1 - 0.4z + 0.04z^2 = 0 \) which are 5 (repeated) are outside the unit circle; or the MA parameters satisfy \( \theta_2 + \theta_1 < 1; \theta_2 - \theta_1 < 1; -1 < \theta_2 < 1 \). Since \( |\theta_2| > 1 \), the process is not causal.

4. B&D, 3.2: 15 points

For any ARMA process, we see that
\[
\begin{align*}
\psi_0 &= 1 \\
\psi_1 &= \theta_1 + \psi_0 \phi_1 \\
\psi_2 &= \theta_2 + \psi_1 \phi_1 + \psi_0 \phi_2 \\
\psi_3 &= \theta_3 + \psi_2 \phi_1 + \psi_1 \phi_2 + \psi_0 \phi_3 \\
\psi_4 &= \theta_4 + \psi_3 \phi_1 + \psi_2 \phi_2 + \psi_1 \phi_3 + \psi_0 \phi_4 \\
\psi_5 &= \theta_5 + \psi_4 \phi_1 + \psi_3 \phi_2 + \psi_2 \phi_3 + \psi_1 \phi_4 + \psi_0 \phi_5, \text{ etc.}
\end{align*}
\]
(a) AR(2) process. The $\psi$ weights are therefore $\psi_0 = 1; \psi_1 = \psi_0 \phi_1; \psi_2 = \psi_1 \phi_1 + \psi_0 \phi_2; \psi_3 = \psi_2 \phi_1 + \psi_1 \phi_2; \psi_4 = \psi_3 \phi_1 + \psi_2 \phi_2; \psi_5 = \psi_4 \phi_1 + \psi_3 \phi_2$, etc. Plugging in $\phi_1 = -0.2$ and $\phi_2 = 0.48$, we can compute the $\psi_j$ weights recursively for $j \geq 0$; substitute these into the form of ACF for a GLP, viz., $\sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$ gives the required result.

Another approach: By multiplying both sides of $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$ by $X_{t-h}$ and taking expectations, we would get the difference equation

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2), h = 1, 2, \ldots$$

while

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2$$

which after some algebra shows that

$$\gamma(0) = \frac{1 - \phi_2 \sigma^2 \phi_1 (1 - \phi_2)}{(1 - \phi_2)^2 - \phi_1^2}$$

(c) ARMA(1,1) process. See Example 3.2.1 in B&D, page 89. Substitute the given coeffs in those formulas.

(d) AR(2) process. Similar to (a).

5. B&D, 3.3 (omit PACF): 15 points

For causal processes in B & D 3.1, we get the first six $\psi$ weights from the general form given under Problem 3 solution:

(a) AR(2) process. $\psi_0 = 1; \psi_1 = -0.2; \psi_2 = 0.52; \psi_3 = -0.2; \psi_4 = 0.2896; \psi_5 = -0.15392$

(c) ARMA(1,1) process. $\psi_0 = 1; \psi_1 = 0.6; \psi_2 = -0.36; \psi_3 = 0.216; \psi_4 = -0.1296; \psi_5 = 0.07776$

(d) $\psi_0 = 1; \psi_1 = -1.8; \psi_2 = 2.43; \psi_3 = -2.916; \psi_4 = 3.2805; \psi_5 = -3.54294$


ACF of the AR(2) process $X_t = 0.8X_{t-2} + Z_t$ is obtained as

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}.$$ 

For this process, we can show that

\[
\begin{align*}
\psi_0 &= 1; \\
\psi_1 &= \psi_0 \phi_1 = 0; \\
\psi_2 &= \psi_1 \phi_1 + \psi_0 \phi_2 = \phi_2 = 0.8; \\
\psi_3 &= \psi_2 \phi_1 + \psi_1 \phi_2 = 0; \\
\psi_4 &= \psi_3 \phi_1 + \psi_2 \phi_2 = (0.8)^2; \text{ etc.}
\end{align*}
\]
Hence,

$$\gamma(h) = \begin{cases} 
\frac{\sigma^2}{(1 - (0.8)^2)} & \text{if } h = 0 \\
\frac{\sigma^2(0.8)^{|h|/2}}{(1 - (0.8)^2)} & \text{if } |h| \text{ is even} \\
0 & \text{if } |h| \text{ is odd}
\end{cases}$$

The ACF is

$$\rho(h) = \begin{cases} 
1 & \text{if } h = 0 \\
(0.8)^{|h|/2} & \text{if } |h| \text{ is even} \\
0 & \text{if } |h| \text{ is odd}
\end{cases}$$

**ST380 Extra Problems:**

1*. B&D 3.5. 20 points
(a) Since $E(X_t) = 0$,

$$E(Y_t) = E(X_t + W_t) = 0$$

and

$$\gamma_Y(h) = \gamma_X(h) + \gamma_W(h)$$

with $\gamma_W(0) = \sigma^2_W$ and $\gamma_W(h) = 0$ for $|h| > 0$, we see that $\{Y_t\}$ is a stationary time series.

The ACVF function of $\{Y_t\}$ has the form

$$\gamma_Y(h) = \sigma^2_Z \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} + \sigma^2_W$$

(b)

$$U_t = \phi(B)Y_t = \phi(B)X_t + \phi(B)W_t$$

$$= \theta(B)Z_t + \phi(B)W_t$$

$$= Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i} + W_t - \sum_{j=1}^{r} \phi_j W_{t-j}$$

so that

$$\gamma_U(h) = \text{Cov}(\sum_{i=0}^{q} \theta_i Z_{t-i} + \sum_{j=0}^{r} \phi_j W_{t-j}, \sum_{i=0}^{q} \theta_i Z_{t-i+h} + \sum_{j=0}^{r} \phi_j W_{t-j+h})$$

For $h > r = \max(p, q)$, $\gamma_U(h) = 0$ since $Z_t$ and $W_t$ are uncorrelated. Hence $\{U_t\}$ is an $r$-correlated process.

2*. B&D 3.6. 10 points
If $X_t = Z_t + \theta Z_{t-1}$, where $Z_t \sim WN(0, \sigma^2)$, we know
\[
\gamma_X(h) = \begin{cases} 
(1 + \theta^2)\sigma^2, & h = 0 \\
\theta\sigma^2, & |h| = 1 \\
0, & |h| > 1
\end{cases}
\]

If \( Y_t = \bar{Z}_t + \frac{1}{\theta}\bar{Z}_{t-1}, \) with \( \bar{Z}_t \sim WN(0, \theta^2\sigma^2), \) we can see that

\[
\gamma_Y(h) = \begin{cases} 
(1 + (1/\theta^2))(\theta^2\sigma^2) = (1 + \theta^2)\sigma^2, & h = 0 \\
(1/\theta)(\theta^2\sigma^2) = \theta\sigma^2, & |h| = 1 \\
0, & |h| > 1
\end{cases}
\]

That is, the ACVF of \( \{X_t\} \) and \( \{Y_t\} \) are same for all \( h. \)