Problem #1 Brockwell and Davis 3.2

From 3.2.14 in Brockwell and Davis, it’s clear that $\alpha_X(1) = \rho_X(1)$. So we only need to compute $\alpha_X(h)$ for $h > 1$.

(a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$. Note this is an AR(2) Process, so by Example 3.2.6 in Brockwell and Davis,

$$\alpha_X(h) = \begin{cases} 
1, & \text{if } h = 0; \\
\rho_X(1), & \text{if } h = 1; \\
0.48, & \text{if } h = 2; \\
0, & \text{if } h > 2.
\end{cases}$$

(c) $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$. This is an ARMA(1,1) process. From example 3.2.1 in Brockwell and Davis, it’s easy to see that

$$\rho_X(h) = \begin{cases} 
1, & \text{if } h = 0; \\
\frac{\theta \sigma^2}{\gamma_X(0)} \phi^{h-1} + \phi^h = 0.768\phi^{h-1} + \phi^h = 0.168\phi^{h-1} = -0.28 \times (-0.6)^h, & \text{if } h \geq 1.
\end{cases}$$

This gives a nice structure of $R_h$ as $R_h = 1.28I_h - 0.28T_h$, where $T_h$ is $h \times h$ Toeplitz matrix with intercorrelation -0.6. This will allow us to compute $\alpha_X(h)$ relative easily. The following are numerically calculated PACF for lag from 2 to 10.

> result
lag  pac
1  2  -0.13277134
2  3  0.10681285
3  4  -0.08692126
4  5  0.07127254
5  6  -0.05873949
6  7  0.04857795
7  8  -0.04026931
8  9  0.03343598
9 10  -0.02779328

(d) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$. Note this is an AR(2) Process, so by Example 3.2.6 in Brockwell and Davis,

$$\alpha_X(h) = \begin{cases} 
1, & \text{if } h = 0; \\
\rho_X(1), & \text{if } h = 1; \\
-0.81, & \text{if } h = 2; \\
0, & \text{if } h > 2.
\end{cases}$$
Problem # 2 Brockwell and Davis 3.4

\[ X_t - 0.8X_{t-2} = Z_t. \] Note this is an AR(2) Process, so by Example 3.2.6 in Brockwell and Davis,

\[ \alpha_X(h) = \begin{cases} 
1, & \text{if } h = 0; \\
\rho_X(1), & \text{if } h = 1; \\
0.8, & \text{if } h = 2; \\
0, & \text{if } h > 2.
\end{cases} \]

Problem # 3 Brockwell and Davis 3.11

We need to show that for the MA(1) Process, \( X_t = Z_t + \theta Z_{t-1} \), \( \alpha_X(2) = -\frac{\theta^2}{1 + \theta^2 + \theta^4} \).

We know that for the above MA(1) process,

\[ \rho_X(h) = \begin{cases} 
1, & \text{if } h = 0; \\
\frac{\theta}{1 + \theta^2}, & \text{if } |h| = 1; \\
0, & \text{if } |h| > 1.
\end{cases} \]

So, at lag 2, we have

\[ R = \left( \begin{array}{cc} 1 & \frac{\theta}{1 + \theta^2} \\ \frac{\theta}{1 + \theta^2} & 1 \end{array} \right), \]

and

\[ \rho = \left( \begin{array}{c} \frac{\theta}{1 + \theta^2} \\ 0 \end{array} \right). \]

So,

\[ \alpha_X(2) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\theta}{1 + \theta^2} \\ \frac{\theta}{1 + \theta^2} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\theta}{1 + \theta^2} \\ 0 \end{pmatrix} = \frac{(1 + \theta^2)^2}{1 + \theta^2 + \theta^4} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\theta}{1 + \theta^2} & -\frac{\theta}{1 + \theta^2} \\ -\frac{\theta}{1 + \theta^2} & 1 \end{pmatrix} \begin{pmatrix} \frac{\theta}{1 + \theta^2} \\ 0 \end{pmatrix} = \frac{(1 + \theta^2)^2}{1 + \theta^2 + \theta^4} \frac{-\theta^2}{(1 + \theta^2)^2} = -\frac{\theta^2}{1 + \theta^2 + \theta^4}. \]