Lectures for STAT280/380 (continued)

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III. Deterministic Time Series Regression Methods

- Structural Decomposition
- Trend Fitting by Polynomial Trend Models
- Trend Fitting by Moving Averages
- Seasonality Fitting using Seasonal Indicators
- Seasonality Fitting using Trigonometric Functions

Suppose $X_1, X_2, \ldots, X_n$ are observed at equi-spaced discrete times $t = 1, \ldots, n$. Think of $\{X_t\}$ as a single realization from an ensemble (a collection of an infinite number of possible realizations) of a stochastic process $\{X_t(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$ where $\mathcal{T}$ denotes a time index set and $\Omega$ denotes a sample space. That is, holding $\omega$ fixed, $\{X_t\}$ is a time series.

$\mathcal{T} : \mathbb{Z}$, or $\mathcal{T} : (-\infty, \infty) = \mathbb{R}$.
$\Omega : 0, 1, \ldots$, or $\Omega : \{0, 1\}$ or $\Omega : \mathbb{R}$. 
If $\mathcal{T} : \mathcal{Z}$ and $\Omega : \mathcal{R}$: continuous-valued, discrete-time series.
If $\mathcal{T} : \mathcal{Z}$ and $\Omega : \{0, 1, \cdots\}$: count-valued, discrete-time series.
If $\mathcal{T} : \mathcal{Z}$ and $\Omega : \{0, 1\}$: binary-valued, discrete-time series.
If $\mathcal{T} : \mathcal{Z}$ and $\Omega : \mathcal{R}$: continuous-valued, continuous-time series.

We will focus only on $\mathcal{T} : \mathcal{Z}$ and $\Omega : \mathcal{R}$: continuous-valued, discrete-time series.
(i) Structural Decomposition

In general, observed time series exhibit features such as trend, and seasonality, in addition to randomness.

**Structural Time Series Model.** Decompose a time series \( X_t \) as the additive model:

\[
X_t = \text{Trend} + \text{Seasonal Component} + \text{Irregular or}
\]

\[
X_t = m_t + S_t + Y_t.
\]

where \( m_t \) denotes the trend component, \( s_t \) denotes the seasonal component, and \( Y_t \) denotes the residual.

This is also called the Classical Decomposition Model.

We will fit \( m_t \) by \( \hat{m}_t \) and fit \( S_t \) by \( \hat{S}_t \); eliminate them from the given time series to get the residual series as \( \hat{Y}_t = X_t - \hat{m}_t - \hat{S}_t \).

Later we can fit a suitable stochastic model to \( \{\hat{Y}_t\} \).
(ii) Trend Fitting by Polynomial Trend Models

Suppose $X_t = m_t + Y_t$, where $EY_t = 0$.

$$m_t = \alpha_0 + \alpha_1 t + \alpha_2 \frac{t^2}{2!} + \cdots + \alpha_k \frac{t^k}{k!}$$

$$= \sum_{j=0}^{k} a_j \frac{t^j}{j!}$$

is a $k$th-order polynomial function of time $t$. $k = 1$: linear trend; $k = 2$: quadratic trend, etc.

These are called Globally Constant Time Series Regression Models because the parameters $\alpha_0, \alpha_1, \cdots, \alpha_k$ are assumed to remain constant over time. We will later study smoothing methods for locally constant time series regression models.
Trend fitting consists of using OLS to fit the time series regression

\[ X_t = \alpha_0 + \alpha_1 Z_{t,1} + \alpha_2 Z_{t,2} + \cdots + \alpha_k Z_{t,k} + \varepsilon_t \]

where \( Z_{t,1} = t, \ Z_{t,2} = \frac{t^2}{2!}, \cdots, \ Z_{t,k} = \frac{t^k}{k!} \)

The fitted values are

\[ \hat{X}_t = \hat{\alpha}_0 + \hat{\alpha}_1 t + \hat{\alpha}_2 \frac{t^2}{2!} + \cdots + \hat{\alpha}_k \frac{t^k}{k!} \]

If the magnitudes of fluctuations in the time series increase as a function of level, a suitable transformation will stabilize the fluctuations. Usually, \( \log(x_t) \) is the transformation employed. Need to determine the order of the polynomial \( k \) - somewhat similar to selection of variables in a multiple regression problem. Look for a model with large \( R^2 \); we may use likelihood based model selection criteria such as AIC; or use forcast evaluation criteria.
See Example 2 for fitting the model \( X_t = a_0 + a_1 t + a_2 t^2 + Y_t \) to data in B & D Example 1.3.4.

(iii) Trend Fitting by Moving Averages

Smooth with a finite moving average filter. An alternate estimate of \( m_t \) is based on a finite moving average filter. Let \( q \) be a nonnegative integer. We use the two-sided moving average of \( \{X_t\} \), viz.,

\[
\hat{m}_t = (2q + 1)^{-1} \sum_{j=-q}^{q} X_{t-j}, \quad q + 1 \leq t \leq n - q
\]

as an estimate of \( m_t \). This formula cannot be used for \( t \leq q \) or \( t > n - q \). To resolve this, we can define \( X_t = X_1 \) for \( t < 1 \) and \( X_t = X_n \) for \( t > n \). This type of filter is called a low-pass filter since it removes high-frequency components from \( \{X_t\} \), leaving the slowly varying trend estimate \( \hat{m}_t \).
Think of the moving average as a linear filter

\[ \hat{m}_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}, \]

where the weights are defined by

\[ a_j = \frac{1}{2q + 1}, \text{ if } -q \leq j \leq q; \text{ and } 0 \text{ otherwise.} \]

Choice of weights \( \{a_j\} \) allows us to design the filter for the data.

An example is Spencer’s 15-point moving average defined by setting weights \( a_j = 0 \) if \(|j| > 7\); \( a_j = a_{-j} \) if \(|j| \leq 7\); and \( a_0 = 74/320; a_1 = 67/320; a_2 = 46/320; a_3 = 21/320; a_4 = 3/320; a_5 = -5/320; a_6 = -6/320; a_7 = -4/320; \) this filter passes polynomials of degree 3 without distortion.
(iv) Seasonality Fitting Using Seasonal Indicators

Assume $X_t = S_t + Y_t$.

Suppose time series exhibit seasonality with seasonal period $s$.
Properties are similar for $X_t, X_{t+s}, X_{t+2s}, \cdots$.
Example: Mortality Data with Seasonal period $s = 12$.
Properties are similar for $X_t, X_{t+12}, X_{t+24}, \cdots$.
See Monthly Accident Data for illustration: 6 years of monthly data.

With quarterly data, $s = 4$, say. Then properties are similar for $X_t, X_{t+4}, X_{t+8}, \cdots$.

Let $X_t = s_t + Y_t$, where $EY_t = 0$, and assume a seasonal period $s$. Set up $s$ indicator functions, viz., $IND_{ti} = 1$ if $t$ corresponds to seasonal period $i$, and 0 otherwise. Using the method of
least squares, fit a seasonal model with seasonal indicators of the form

$$s_t = \sum_{i=1}^{s} \delta_i IND_{ti}.$$  

It is also possible to combine the polynomial trend to this model to fit $X_t = m_t + s_t + Y_t$, where $m_t$ was defined earlier.

See Example 3 for fitting the model with a linear trend and seasonal indicators to data on monthly accidental deaths shown in B & D Example 1.1.3.

(v) Seasonality Fitting using Trigonometric Functions. Let $X_t = s_t + Y_t$, where $EY_t = 0$, and assume a seasonal period $d$. Let

$$s_t = a_0 + \sum_{j=1}^{k}[a_j \cos(2\pi jt/d) + b_j \sin(2\pi jt/d)].$$
Number of harmonics is $k$. The model, including a suitable polynomial trend is fit using least squares.

See Example 3 for fitting the model with a linear trend and trigonometric functions to data on monthly accidental deaths shown in B & D Example 1.1.3.

**(vi) Steps in Structural Time Series Modeling.**

Step 1. Check the time series plot for evidence of (a) trend, (b) seasonality, (c) change in behavior (such as level change, or scale change), (d) extreme observations.

Step 2. If the data shows evidence of trend $m_t$ and/or seasonality $s_t$, estimate them by the method of least squares.
Step 3. In an additive structural model, \( X_t = m_t + s_t + Y_t \), we obtain the model residuals as

\[
\hat{Y}_t = X_t - \hat{m}_t - \hat{s}_t,
\]

Step 4. If the residuals do not exhibit any further pattern, then the best fit for \( x_t \) is \( \hat{m}_t + \hat{s}_t \). If not, the residuals will be analyzed for stochastic pattern using time series techniques, and a suitable time series model will be fit to the residuals. Based on this model, forecasts of future unknown residuals will be obtained and used along with \( \hat{m}_t + \hat{s}_t \) to obtain forecasts of \( X_t \).