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ECE Workshop May 11 2007
Moment Generating Function (MGF)

The MGF of a random variable $Y$ is defined as

$$m(t) = E[\exp(tY)] =
\begin{cases}
\int \exp(tY)f(y)\,dy & \text{if } Y \text{ is continuous} \\
\sum_y \exp(tY)p(y) & \text{if } Y \text{ is discrete}
\end{cases}$$

We say the MGF of $Y$ exists if there exists a positive constant $b$ such that $m(t)$ is finite for $|t| \leq b$. 
Examples:

**Binomial Distribution** $Y \sim Bin(n, p)$:

$m(t) = (pe^t + (1 - p))^n$.

**Poisson Distribution** $Y \sim Poisson(\lambda)$:

$m(t) = \exp[\lambda(e^t - 1)]$.

**Normal Distribution** $Y \sim N(\mu, \sigma^2)$:

$m(t) = \exp(\mu t + \sigma^2 t^2 / 2)$.
Some Useful Properties of MGF:

1. Suppose $Y$ is a $N(\mu, \sigma^2)$ random variable, so that

$$m_Y(t) = \exp(\mu t + \sigma^2 t^2 / 2)$$

Then, MGF of $Y - \mu$ is

$$m_{Y-\mu}(t) = E[e^{t(Y-\mu)}] = e^{-t\mu} E[e^{tY}]$$

$$= \exp(-\mu t) \times \exp(\mu t + \sigma^2 t^2 / 2)$$

$$= \exp(\sigma^2 t^2 / 2)$$

This is the form of MGF of a $N(0, \sigma^2)$ variable.
2. Suppose $Y$ is a $N(\mu, \sigma^2)$ random variable, so that

$$m_Y(t) = \exp(\mu t + \sigma^2 t^2/2)$$

Let $Z = (Y - \mu)/\sigma$. Then

$$m_Z(t) = E[e^{tZ}] = E[t((Y - \mu)/\sigma)]$$
$$= E[(t/\sigma)(Y - \mu)] = m_{Y-\mu}(t/\sigma)$$
$$= \exp[(t/\sigma)^2(\sigma^2/2)] = \exp(t^2/2)$$

This is the MGF of a $N(0, 1)$ variable.
3. Let $Y_1, \ldots, Y_n$ be independent random variables with respective MGF’s $m_{Y_1}(t), \ldots, m_{Y_n}(t)$. Define a random variable $U = Y_1 + \cdots + Y_n$. Then

$$m_U(t) = E[\exp(tU)] = E[\exp(t(Y_1 + \cdots + Y_n))]$$

$$= E[\exp(tY_1) \times \cdots \times \exp(tY_n)]$$

$$= E[\exp(tY_1)] \times \cdots \times E[\exp(tY_n)]$$

$$= m_{Y_1}(t) \times \cdots \times m_{Y_n}(t)$$
4. Let $Y_1, \ldots, Y_n$ be independent Normal random variables such that $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$ for $i = 1, \ldots, n$. The MGF of $Y_i$ is

$$m_{Y_i}(t) = \exp(\mu_i t + \sigma_i^2 t^2 / 2)$$

Let $c_1, \ldots, c_n$ be real constants. MGF of $c_i Y_i$ is

$$m_{c_i Y_i}(t) = E[t c_i Y_i] = m_{Y_i}(c_i t) = \exp(\mu_i c_i t + \sigma_i^2 c_i^2 t^2 / 2)$$

Define a new random variable

$$U = \sum_{i=1}^{n} c_i Y_i = c_1 Y_1 + \cdots + c_n Y_n$$

Then, MGF of $U$ is

$$m_U(t) = m_{c_1 Y_1}(t) \times \cdots \times m_{c_n Y_n}(t)$$

$$= \prod_{i=1}^{n} \exp[\mu_i c_i t + \sigma_i^2 c_i^2 t^2 / 2]$$

$$= \exp[t \sum_{i=1}^{n} c_i \mu_i + \frac{t^2}{2} \sum_{i=1}^{n} c_i^2 \sigma_i^2]$$
Method of Moment Generating Function to find Distribution of a New Random Variable

Find the probability distribution of a function of random variables $Y_1, \ldots, Y_n$ based on the “uniqueness theorem”.

Uniqueness Theorem: Let $m_X(t)$ and $m_Y(t)$ respectively denote the MGF’s of random variables $X$ and $Y$. If both $m_X(t)$ and $m_Y(t)$ exist, and if $m_X(t) = m_Y(t)$ for all values of $t$, then the two random variables $X$ and $Y$ have the same probability distribution.
Summary of Method

Let $U$ be a random variable which is a function of random variables $Y_1, \cdots, Y_n$. We seek the distribution of $U$.

Step 1: Find the MGF $m_U(t)$ of the r.v. $U$.

Step 2: Compare $m_U(t)$ with other well-known MGF forms of known probability distributions. If $m_U(t) = m_V(t)$ for all values of $t$, then by the Uniqueness theorem, $U$ and $V$ have identical probability distributions.
Distributions Related to the Normal Distribution

Suppose $Y$ is a $N(\mu, \sigma^2)$ random variable. Then, by the Uniqueness Theorem,

- $Y - \mu$ has a $N(0, \sigma^2)$ distribution.

- $Z = (Y - \mu)/\sigma$ has a $N(0, 1)$ distribution.
Let $Y_1, \ldots, Y_n$ be independent Normal random variables such that $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$ for $i = 1, \ldots, n$. Let $c_1, \ldots, c_n$ be real constants. Let

$$U = \sum_{i=1}^{n} c_i Y_i = c_1 Y_1 + \cdots + c_n Y_n$$

By the Uniqueness Theorem, $U$ is distributed as a Normal random variable with mean

$$E(U) = \sum_{i=1}^{n} c_i \mu_i = c_1 \mu_1 + \cdots + c_n \mu_n$$

and variance

$$Var(U) = \sum_{i=1}^{n} c_i^2 \sigma_i^2 = c_1^2 \sigma_1^2 + \cdots + c_n^2 \sigma_n^2$$
• Sampling Distribution of $\bar{Y}$

Let $Y_1, \ldots, Y_n$: random sample from a $N(\mu, \sigma^2)$ distribution.

Let $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$. Compare to $U$ with $c_i = 1/n$ for $i = 1, \ldots, n$.

Then, $\bar{Y} \sim N(\mu_{\bar{Y}} = \mu, \sigma^2_{\bar{Y}} = \sigma^2/n)$.

Hence,

$$Z = \frac{\bar{Y} - \mu_{\bar{Y}}}{\sigma_{\bar{Y}}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \sim N(0, 1).$$

This gives the basis for inference about $\mu$ from a normal population with known $\sigma^2$. 

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Example: Suppose a bottling machine can discharge an average of $\mu$ oz. per bottle. The amount of fill dispensed by the machine has been observed to be normally distributed with $\sigma = 1.0$ oz.

a) A sample of $n = 9$ filled bottles is randomly selected on a given day from the machine at the same settings and the filling is measured for each. Find the probability that at that setting, the sample mean will be within 0.3 oz. of the true mean.

Ans: 0.6318
b) Find the sample size if we wish $\bar{Y}$ to be within 0.3 oz of $\mu$ with probability 0.95.

Ans: $n$ must exceed 42.68; so set $n = 43$. 

Let $Z \sim N(0, 1)$. So $m_Z(t) = \exp(t^2/2)$.

Using some special properties of integrals, we may show that

$$m_{Z^2}(t) = (1 - 2t)^{-1/2}$$

This is the MGF of a Chi-square distribution with $\nu = 1$ d.f. Hence $Z^2$ has a $\chi^2_1$ distribution.
Let \( Y_1, \ldots, Y_n \) be independent \( N(\mu_i, \sigma_i^2) \) distribution for \( i = 1, \ldots, n \). Let \( Z_i = (Y_i - \mu_i)/\sigma_i \). As before,

\[
m_{Z_i^2}(t) = (1 - 2t)^{-1/2}
\]

Let \( V = \sum_{i=1}^{n} Z_i^2 \). Then

\[
m_V(t) = m_{Z_1^2}(t) \times \cdots \times m_{Z_n^2}(t) = (1 - 2t)^{-1/2} \times \cdots \times (1 - 2t)^{-1/2} = (1 - 2t)^{-n/2}
\]

By the Uniqueness theorem, \( V \sim \chi_n^2 \).
Example: Suppose $Z_1, \cdots, Z_6$ is a random sample from a $N(0,1)$ distribution. Find $a$ s.t. $P(\sum_{i=1}^{6} Z_i^2 \leq a) = 0.95$. Hint: Use the table on percentage points from a $\chi^2$ distribution.

Ans: $a = 12.5916$
• Sampling Distribution of $S^2$

Let $Y_1, \ldots, Y_n$: random sample from a $N(\mu, \sigma^2)$
distribution.

Let $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$. Then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sim \chi^2_{n-1}$$

Note that here $\mu$ is not known and is “estimated” by $\bar{Y}$. 

Example: In the bottling example, suppose we wish to find numbers $a_1$ and $a_2$ s.t. based on $n = 10$ samples,

$$P(a_1 \leq S^2 \leq a_2) = 0.9$$

Ans: $a_1 = 0.369$ and $a_2 = 1.880$. 


Let $Y_1, \cdots, Y_n$: random sample from a $N(\mu, \sigma^2)$ distribution. We saw that

$$Z = \frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \sim N(0, 1)$$

Also,

$$W = (n - 1)S^2/\sigma^2 \sim \chi^2_{\nu=n-1} d.f.$$ 

Can show that $Z$ and $W$ are independently distributed (i.e., $\bar{Y}$ and $S^2$ are independently distributed).

The random variable

$$T = \frac{Z}{\sqrt{W/\nu}}$$

$$= \frac{\sqrt{n}(\bar{Y} - \mu)/\sigma}{\sqrt{[(n - 1)S^2/\sigma^2]/(n - 1)}}$$

$$= \frac{\sqrt{n}(\bar{Y} - \mu)}{S}$$

has (by definition) a Student's-$t$ distribution with $(n - 1)$ d.f.
Example: The tensile strength of a type of wire is distributed as $N(\mu, \sigma^2)$. Let $Y_1, \cdots, Y_6$ denote the tensile strength measured from $n = 6$ pieces of wire randomly selected from a large roll. What is the approximate probability that $\bar{Y}$ will be within $2S/\sqrt{n}$ of $\mu$?

Ans: Approx. 0.9
Let $Y_{1,1}, \ldots, Y_{1,n_1}$: random sample from a $N(\mu_1, \sigma^2_1)$ distribution.

Let $\bar{Y}_1$ and $S^2_1$: sample mean and variance.

Let $Y_{2,1}, \ldots, Y_{2,n_2}$: random sample from a $N(\mu_2, \sigma^2_2)$ distribution.

Let $\bar{Y}_2$ and $S^2_2$: sample mean and variance.

The variable

$$F = \frac{S^2_1/\sigma^2_1}{S^2_2/\sigma^2_2} = \frac{\sigma^2_2 S^2_1}{\sigma^2_1 S^2_2} \sim F(n_1-1),(n_2-1)$$
Example: Suppose we take independent samples of size $n_1 = 6$ and $n_2 = 10$ from two normal populations with $\sigma_1^2 = \sigma_2^2$. Find a constant $a$ s.t.

$$P\left(\frac{S_1^2}{S_2^2} \leq a\right) = 0.95$$

Ans: 3.48
Central Limit Theorem and Large Sample Inference

Let $Y_1, \ldots, Y_n$: random sample from a $N(\mu, \sigma^2)$ distribution, with $\sigma^2 < \infty$.

Let $\bar{Y}$ and $S^2$: sample mean and variance. Let

$$U_n = \sqrt{n} \frac{\bar{Y} - \mu}{\sigma}.$$

As $n \to \infty$, the distribution function of $U_n$ converges to a $N(0, 1)$ distribution. Or

$$U_n = \sqrt{n} \frac{\bar{Y} - \mu}{s}.$$
Example: Suppose service times for customers at a checkout counter at a store are independent random variables with mean 1.5 mins and variance 1.0 min. Approximately, what is the probability that 100 customers may be served in less than 2 hours of total service time?

Ans: 0.0013
Large Sample Distributional Properties

- \( \bar{Y} \sim N(\mu, \sigma^2/n) \)

- \( \bar{Y}_1 - \bar{Y}_2 \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}) \)  
  (assuming independence of two samples)

- \( \hat{p} \sim N(p, pq/n) \)

- \( \hat{p}_1 - \hat{p}_2 \sim N(p_1 - p_2, \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}) \)  
  (assuming independence of two samples)
Large Sample C.I. for a parameter $\theta$

Based on point estimate $\hat{\theta}$ with standard error $\sigma_{\hat{\theta}}$.

The statistic

$$Z = \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \sim N(0, 1)$$

Hence,

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha, \text{ i.e.,}$$

$$P(-z_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \leq z_{\alpha/2}) = 1 - \alpha, \text{ i.e.,}$$

$$P(-z_{\alpha/2} \sigma_{\hat{\theta}} \leq \hat{\theta} - \theta \leq z_{\alpha/2} \sigma_{\hat{\theta}}) = 1 - \alpha, \text{ i.e.,}$$

$$P(-\hat{\theta} - z_{\alpha/2} \sigma_{\hat{\theta}} \leq -\theta \leq -\hat{\theta} + z_{\alpha/2} \sigma_{\hat{\theta}}) = 1 - \alpha$$

$$P(\hat{\theta} - z_{\alpha/2} \sigma_{\hat{\theta}} \leq -\theta \leq \hat{\theta} + z_{\alpha/2} \sigma_{\hat{\theta}}) = 1 - \alpha$$
For $\mu$:
Pivotal quantity: $Z = (\bar{Y} - \mu)/\sigma_{\bar{Y}}$.

$100(1 - \alpha)\%$ C.I.: $\bar{Y} \pm z_{\alpha/2}s/\sqrt{n}$

For $p$:
Pivotal quantity: $Z = (\hat{p} - p)/\sigma_{\hat{p}}$.

$100(1 - \alpha)\%$ C.I.: $\hat{p} \pm z_{\alpha/2}\sqrt{\hat{p}\hat{q}/n}$

For $\mu_1 - \mu_2$:
Pivotal quantity: $Z = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma^2_1/n_1 + \sigma^2_2/n_2}}$

$100(1-\alpha)\%$ C.I.: $\bar{Y}_1 - \bar{Y}_2 \pm z_{\alpha/2}\sqrt{s^2_1/n_1 + s^2_2/n_2}$

For $p_1 - p_2$:
Pivotal quantity:

$Z = \frac{[\hat{p}_1 - \hat{p}_2] - (p_1 - p_2)}{\sqrt{\hat{p}_1\hat{q}_1/n_1 + \hat{p}_2\hat{q}_2/n_2}}$

$100(1-\alpha)\%$ C.I.: $\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\hat{p}_1\hat{q}_1/n_1 + \hat{p}_2\hat{q}_2/n_2}$
Example: The shopping times for \( n = 64 \) randomly chosen customers at a local store had an average of 33 mins and variance of 256 mins. Construct a 90\% C.I. for \( \mu \).

Ans: (29.71, 36.29)
Example: Two brands of refrigerators, Brand A and Brand B are each guaranteed for 1 year. In a random sample of $n_1 = 50$ units of Brand A, 12 failed within 1 year. In an independent random sample of $n_2 = 60$ units of Brand B, 12 again failed within 1 year. With 98% confidence, estimate $p_1 - p_2$.

Ans: (-0.1451, 0.2251)
Small Sample C.I.

\[ T = \frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1} \]

Pivotal quantity.

\[ P\left(-t_{\alpha/2} \leq T \leq t_{\alpha/2}\right) \]

so that 100(1 - \alpha)% C.I. is

\[ \bar{Y} \pm t_{\alpha/2, n-1} S/\sqrt{n} \]
Example: A new gunpowder developed by a manufacturer was tested in 8 shells. The sample mean and standard deviation of the muzzle velocities (feet per sec) were $\bar{y} = 2959$ and $s = 39$. Find a 95% C.I. for $\mu$. What distributional assumption is needed?

Ans: $2959 \pm 32.7$
Small sample $\mu_1 - \mu_2$:

Pivotal quantity:

$$T = \frac{(\bar{Y}_1 - \bar{Y}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{1/n_1 + 1/n_2}}$$

which has a $t_{n_1+n_2-2}$ d.f., and

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$100(1 - \alpha)%$ C.I. is:

$$(\bar{Y}_1 - \bar{Y}_2) \pm t_{\alpha/2, n_1+n_2-2} \times S_p \sqrt{1/n_1 + 1/n_2}$$
Example: To compare a new method of employee training with a standard procedure on time taken to assemble a device, 9 employees were trained for a 3-week period under the new method, and an independent set of 9 employees under the standard method. $\bar{y}_1 = 35.22; S^2_1 = 24.445; \bar{y}_2 = 31.56; S^2_2 = 20.027$; Construct a 95% C.I. for $\mu_1 - \mu_2$.

Ans: $3.66 \pm 4.71$
Distributions Underlying Hypothesis Testing for developing the
• Rejection Region and
• Power of the Test

A Discussion.
References