The Annals of Applied Probability
2015, Vol. 25, No. 3, 1513–1539
DOI: 10.1214/14-AAP1029

STRONG RENEWAL THEOREMS WITH INFINITE MEAN BEYOND LOCAL LARGE DEVIATIONS

BY ZHIYI CHI

University of Connecticut

Let F be a distribution function on the line in the domain of attraction of a stable law with exponent $\alpha \in (0, 1/2]$. We establish the strong renewal theorem for a random walk $S_1, S_2, \dots$ with step distribution F, by extending the large deviations approach in Doney [*Probab. Theory Related Fileds* **107** (1997) 451–465]. This is done by introducing conditions on F that in general rule out local large deviations bounds of the type $\mathbb{P}\{S_n \in (x, x+h]\} = O(n)\overline{F}(x)/x$, hence are significantly weaker than the boundedness condition in Doney (1997). We also give applications of the results on ladder height processes and infinitely divisible distributions.

1. Introduction. Let $X, X_1, X_2, \dots$ be i.i.d. real-valued random variables with distribution function F. Denote $S_n = \sum_{i=1}^n X_i$. This article concerns the asymptotic of

$$U(x+I) := \sum_{n=1}^{\infty} \mathbb{P}\{S_n \in x+I\} \qquad \text{as } x \to \infty, \tag{1.1}$$

under certain tail conditions on F, where $I = (0, h]$ with $h \in (0, \infty)$. Specifically, denoting by $\mathcal{R}_\alpha$ the class of functions that are regularly varying at ∞ with exponent α and $\overline{F}(x) = 1 - F(x) = \mathbb{P}\{X > x\}$, the first condition is

$$\overline{F}(x) \sim 1/A(x) \qquad \text{as } x \to \infty \text{ with } A \in \mathcal{R}_\alpha, \alpha \in (0, 1). \tag{1.2}$$

By (1.2), $p^+ := \mathbb{P}\{X > 0\} > 0$. The second condition is the tail ratio condition

$$r_F := \lim_{x\to\infty} \{F(-x)/\overline{F}(x)\} \qquad \text{exists and is finite.} \tag{1.3}$$

Actually, we often only need the following weaker tail ratio condition:

$$\limsup_{x\to\infty} \{F(-x)/\overline{F}(x)\} < r < \infty. \tag{1.4}$$

There are several well-known works on the strong renewal theorem (SRT) for S_n, that is, the nontrivial limit of $x\overline{F}(x)U(x+I)$ as $x \to \infty$ with $0 < h < \infty$; see [4, 9, 20] for the arithmetic case and [7] for the nonarithmetic case. The definitions of (non)arithmetic distributions and the related (non)lattice distributions are

Received June 2013; revised March 2014.
MSC2010 subject classifications. 60K05, 60F10.
Key words and phrases. Renewal, regular variation, ladder height process, infinitely divisible, large deviations, concentration function.

given in Section 2. While the SRT always holds for $\alpha \in (1/2, 1]$ in the arithmetic case as well as in the nonarithmetic case with the extra condition $\mathbb{P}\{X \geq 0\} = 1$, there are examples where it fails to hold for $\alpha \in (0, 1/2]$; see [20], and also [9] for explanations. For the arithmetic case, a well-known condition that leads to the SRT for all $\alpha \in (0, 1/2]$ is

$$\sup_{n \geq 0} \{ n\mathbb{P}\{X = n\} / \overline{F}(n) \} < \infty,$$

provided X is integer-valued. Under the condition, the SRT was established for $1/4 < \alpha \leq 1/2$ in [20]. The general arithmetic case remained open until [4], which took a different approach from previous efforts. The core of the argument in [4] is an estimate of local large deviations (LLD) for the events $\{S_n \in x + I\}$ as $x \to \infty$. Once these estimates are established, the rest of the proof is basically an application of the local limit theorems (LLTs) ([1], Theorem 8.4.1–2) [7, 9, 20]. Recently, it was shown [18] that, for the nonlattice case, if

$$\sup_{x \geq 0} \omega_I(x) < \infty, \tag{1.5}$$

where for $I \subset \mathbb{R}$,

$$\omega_I(x) = x\mathbb{P}\{X \in x + I\} / \overline{F}(x), \tag{1.6}$$

then a much simpler argument than the one in [4] can be used to get the same type of LLD bound, which then leads to the SRT.

However, condition (1.5) can be restrictive. As an example, let F be supported on $[1, \infty)$ and have piecewise constant density $f(x) \propto h(x)$, such that

$$h(x) = \begin{cases} n^{-\alpha-1}, & n \leq x < n+1, n \in \mathbb{N} \setminus \{2^k, k \geq 1\}, \\ kn^{-\alpha-1}, & n = 2^k \leq x < n+1, k \in \mathbb{N}. \end{cases}$$

Then $\overline{F}(x) \sim C/x^{-\alpha}$ as $x \to \infty$, where $C, C', \ldots$ denote constants. Set $I = (0, 1]$. Since $\omega_I(x) \sim C' \ln x$ for $x \in [2^k, 2^k + 1)$, (1.5) does not hold. On the other hand, the set of x with large $\omega_I(x)$ has low density of order $O(\ln x / x)$, while the large values of $\omega_I(x)$ increases slowly at order $O(\ln x)$ as $x \to \infty$. Thus it is reasonable to wonder if the SRT should still hold.

To handle similar situations as the example, one way is to control the aggregate effect of large values of $\omega_I(x)$. We therefore define the function

$$K(x, T) = K(x, T; I) = \int_0^x [\omega_I(y) - T]^+ \, dy, \tag{1.7}$$

where $c^\pm = \max(\pm c, 0)$ for $c \in \mathbb{R}$ and $T > 0$ is a parameter. We will show that, for example, if $X > 0$ and $\alpha \in (0, 1/2)$, and if for some $T > 0$,

$$K(x, T) = o(A(x)^2), \tag{1.8}$$

then the SRT holds for S_n. Since (1.5) implies $K(x, T) \equiv 0$ if $T > 0$ is large enough, it is a special case of (1.8). In the above example, since $K(x, T) =$

$O((\ln x)^2)$ for large $T > 0$, the SRT holds as well. Notice that if (1.8) holds for one $h \in (0, \infty)$, it holds for all $h \in (0, \infty)$. As will be seen, the condition can be further relaxed.

There has been constant interest in the large deviations of sums of random variables with regularly varying distributions and infinite mean ($0 < \alpha < 1$), most notably in the "big-jump" domain; see [3–5, 11, 16] and references therein. The theme of this line of research is to identify the domain of large x, such that the event $S_n \in x + I$ with $a_n \ll x$ and $0 < h \le \infty$ is mainly due to a single large value among $X_1, \dots, X_n$. Here a_n are constants such that S_n/a_n is tight; see the definition of a_n in Section 2. The local version of this type of large deviations, with $h < \infty$ as opposed to $h = \infty$ in the global version, requires more elaborate conditions on $\mathbb{P}\{X \in x + I\}$, and it seems that none of the conditions in the current literature allows occasional large values of $\omega_I(x)$. As shown in [4, 18], to establish the SRT, the precise LLD $\mathbb{P}\{S_n \in x + I\} \sim n\mathbb{P}\{X \in x + I\}$ is unnecessary, and instead $\mathbb{P}\{S_n \in x + I\} = O(n)\mathbb{P}\{X \in x + I\}$ or even $\mathbb{P}\{S_n \in x + I\} = O(n)\overline{F}(x)/x$ can be the starting point. Our results implies the latter is not necessary either; in the above example, for each fixed large n, $\limsup_{x\to\infty} \mathbb{P}\{S_n \in x + I\}/[n\overline{F}(x)/x] = \infty$, because as $k \to \infty$,

$$\begin{aligned}\mathbb{P}\{S_n \in [2^k + a_n, 2^k + a_n + 1)\} &\\ \ge n\mathbb{P}\{X \in [2^k, 2^k + 1/2)\}\mathbb{P}\{S_{n-1} \in [a_n, a_n + 1/2)\} &\\ \sim cna_n^{-1}k/2^{k(\alpha+1)} &\end{aligned}$$

and $\overline{F}(2^k + a_n)/(2^k + a_n) \sim c'/2^{k(\alpha+1)}$, where $c, c' > 0$ are constants. On the other hand, as in [4], we still need certain estimates of the Lévy concentration function of S_n [13]. These are systematically furnished by the analysis on small-step sequences in [3].

As an application, we will consider the ladder height process of S_n. Because F is the basic information, it is of interest to find conditions on K that yield the SRT for the ladder height process. It is known that under certain conditions, the step distribution of the ladder height process is in the domain of attraction of stable law with exponent $\alpha\varrho$, where $0 \le \varrho \le 1$ is the positivity parameter of the limiting stable distribution associated with S_n [14]. We will show that, if $K(x, T) = O(A(x)^{2c})$ for some $c \in [0, \varrho)$, then the SRT holds for the ladder process. Note that since the ladder steps are nonnegative, due to the results in [7, 9], only the case where $\alpha\varrho \le 1/2$ needs to be considered.

As another application, we will also consider the case where X is infinitely divisible. Since the Lévy measure ν of X is typically much easier to specify than its distribution function F, a natural question is whether similar conditions on $K(x, T)$ can be found for ν that lead to the SRT. This question turns out to have a positive answer. Naturally, it is more interesting and important to study the SRT for Lévy processes under a similar setting. However, this is beyond the scope of the paper.

The main results are stated in Section 2 and their proofs are given in Section 3.

2. Main results. Since other than (1.2), there are no constraints on $A(x)$, we shall always assume without loss of generality that it is continuous and strictly increasing on $[0,\infty)$ with $A(0) > 0$, such that for $x \gg 1$, $A(x) = x^\alpha \exp\{\int_1^x \varepsilon(v)\,\mathrm{d}v/v\}$, where $\varepsilon(v)$ is bounded and continuous, and $\varepsilon(v) \to 0$ as $v \to \infty$ ([12], Theorem IV2.2). Then

$$A^{-1}(x) = x^{1/\alpha}\beta(x) \tag{2.1}$$

is continuous and strictly increasing, with $\beta \in \mathcal{R}_0$ ([1], Theorem 1.5.12). Furthermore, by $A'(x) \sim \alpha A(x)/x$, $(A^{-1})'(x) \sim A^{-1}(x)/(\alpha x)$. Denote $a_n = A^{-1}(n)$. Then, as $n \to \infty$, $a_n \to \infty$ and $n(1 - a_n/a_{n+1}) = 1/\alpha + o(1)$.

Under conditions (1.2) and (1.3), $S_n/a_n \xrightarrow{D} \zeta$, where ζ is a stable random variable such that ([2], pages 207–213)

$$\mathbb{E}[e^{\mathrm{i}\theta\zeta}] = \exp\left\{\int (e^{\mathrm{i}\theta x} - 1)\lambda(x)\,\mathrm{d}x\right\}$$
$$\text{with } \lambda(x) = \mathbf{1}\{x > 0\}x^{-\alpha-1} + r_F\mathbf{1}\{x < 0\}|x|^{-\alpha-1}.$$

Letting $\gamma = (1 - r_F)/(1 + r_F)$, the positivity parameter $\varrho = \mathbb{P}\{\zeta > 0\}$ of ζ is equal to $1/2 + (\pi\alpha)^{-1}\tan^{-1}(\gamma\tan(\pi\alpha/2))$ ([1], page 380). Let g denote the density of ζ.

Henceforth, F is said to be arithmetic (resp., lattice), if there is $d > 0$, such that its support is contained in $d\mathbb{Z}$ (resp., $a + d\mathbb{Z}$ for some $0 \le a < d$). In either case, the span of F is the largest such d. F is said to be nonarithmetic (resp., nonlattice) if it is not arithmetic (resp., not lattice). A lattice distribution can be nonarithmetic. Indeed, provided the span of the distribution is d, the distribution is nonarithmetic $\Longleftrightarrow$ its support is contained in $a + d\mathbb{Z}$ for some $a > 0$ with a/d being an irrational number.

We shall always assume $h > 0$ is fixed. Since it is well-known that for $\alpha \in (1/2, 1)$, the SRT holds if (1) F is nonarithmetic and concentrated in $[0,\infty)$ [7], or (2) F is arithmetic [9, 20], we shall only consider $\alpha \in (0, 1/2]$.

The main results of this section are obtained under the following:

ASSUMPTION 1. There exist a function L and a constant $T_0 > 0$ such that, letting $\theta = 1/\alpha - 1$, the following hold:

(a) $L \in \mathcal{R}_c$ for some $c \in [0,\alpha]$ and is nondecreasing. If $p^+ = 1$, then $L(x) \to \infty$. If $p^+ \in (0,1)$, then $L(x)/\ln x \to \infty$. Furthermore,

$$x\overline{F}(x) \sum_{n \le L(x)} \mathbb{P}\{S_n \in x + I\} \to 0, \qquad x \to \infty. \tag{2.2}$$

(b) If $\alpha \in (0, 1/2)$, then

$$K(x, T_0) = o\left(\frac{A(x)^2}{u_\theta(x)}\right), \qquad \text{where } u_\theta(x) = \sum_{n \ge L(x)} \frac{n^{-\theta}}{\beta(n)}; \tag{2.3}$$

(c) If $\alpha = 1/2$, then

$$K(x, T_0) = \begin{cases} O\left(\dfrac{A(x)^2}{\tilde{u}(x)}\right), & \dfrac{\tilde{u}(x)}{\tilde{u}(L(x))} \to 1, \\ o\left(\dfrac{A(x)^2}{\tilde{u}(x)}\right), & \text{else}, \end{cases} \tag{2.4}$$

$$\text{where } \tilde{u}(x) = \int_1^{A(x)} \frac{y^{-1}\,\mathrm{d}y}{\beta(y)}.$$

THEOREM 2.1. *Let $\alpha \in (0, 1/2]$ and (1.2)–(1.3) hold. Then Assumption 1 implies the SRT*

$$\lim_{x\to\infty} x\overline{F}(x)U(x+I) = h\Lambda_F \qquad \textit{with } \Lambda_F = \alpha \int_0^\infty x^{-\alpha} g(x)\,\mathrm{d}x, \tag{2.5}$$

where $h > 0$ is arbitrary if F is nonarithmetic, and is the span of F otherwise.

REMARK.

(1) If $r_F = 0$ in (1.3), then $\Lambda_F = \sin(\pi\alpha)/\pi$; see [7].

(2) Under Assumptions 1(a) and (b), $u_\theta \in \mathcal{R}_{c(2-1/\alpha)}$. Since $\theta = 1/\alpha - 1 > 1$, if $c > 0$, then clearly the order of the bound in (2.3) is strictly higher than $A(x)^2$. If $c = 0$, then by $L(x) \to \infty$, $u_\theta(x) = o(1)$, so the bound in (2.3) is still strictly higher than $A(x)^2$.

(3) In Assumption 1(c), $\tilde{u}(x)$ is increasing in $x > 0$. Also, either $\tilde{u} \in \mathcal{R}_0$ or $\tilde{u}(x)$ converges to a finite number as $x \to \infty$.

(4) The integral conditions in (2.3) and (2.4) also imply some "hard" upper limits to $\omega_I(x)$. Indeed, since uniformly for $t \in [0, h]$,

$$\begin{aligned} &\omega_I(x-t) + \omega_I(x+h-t) \\ &\quad \sim \frac{x}{\overline{F}(x)}\left[\mathbb{P}\{X \in x-t+I\} + \mathbb{P}\{X \in x+h-t+I\}\right] \ge \omega_I(x) \end{aligned}$$

as $x \to \infty$, if $\omega_I(x_n) \to \infty$ for a sequence $x_n \to \infty$, then for any $T > 0$,

$$\begin{aligned} K(x_n+h, T) &\ge \int_0^h \left(\left[\omega_I(x_n-t) - T\right]^+ + \left[\omega_I(x_n+h-t) - T\right]^+\right)\mathrm{d}t \\ &\ge h\left[\omega_I(x_n) - 2T\right]^+. \end{aligned} \tag{2.6}$$

Therefore, the bound in (2.3) or (2.4) applies to $\omega_I(x)$ as well.

It can be shown that if the SRT holds, then (2.2) holds for any $L(x) = o(A(x))$; see the Appendix. The question is, before validating the SRT for F, whether one can find L so that (2.2) holds, and if so, how fast L can grow? It is easy to see that if

$$x\overline{F}(x)\mathbb{P}\{X \in x+I\} \to 0 \quad \text{or, equivalently} \quad \omega_I(x) = o\big(A(x)^2\big), \tag{2.7}$$

then (2.2) holds provided L grows slowly enough, and so Assumption 1(a) is satisfied if $p^+ = 1$. Also note that by (2.6), $\omega_I(x) = O(K(x, T_0))$. Then the next result is immediate.

COROLLARY 2.2. *Let $p^+ = 1$ and (1.2)–(1.3) hold. Let $\alpha \in (0, 1/2)$ or $\alpha = 1/2$ and $\tilde{u}(\infty) < \infty$. If (2.7) holds and $K(x, T_0) = O(A(x)^2)$, in particular, if $K(x, T_0) = o(A(x)^2)$, then the SRT holds for U.*

EXAMPLE. In [20], it is shown that if X only takes values in $\mathbb{N}$, such that

$$\mathbb{P}\{X = n\} = \begin{cases} Cn^{-3/2} \ln n, & n \neq 2^k \text{ for some } k \in \{0\} \cup \mathbb{N}, \\ Cn^{-1/2}/(\ln n)^q, & \text{otherwise, with } q = 1, \end{cases}$$

where $C = C(q) > 0$ is a constant that may change from line to line, then (2.7) does not hold, and hence the SRT fails to hold. We show that if $q \geq 2$, then the SRT holds. First, as in [20], $\overline{F}(x) \sim x^{-1/2} \ln x$, where $C > 0$ is a constant. Then $\alpha = 1/2$, (2.7) holds, and we can set $A(x) = C\sqrt{x}/\ln x$. Since X is aperiodic with support $\{0\} \cup \mathbb{N}$, $h = 1$. It follows that $A^{-1}(x) = x^2 \beta(x) \sim C(x \ln x)^2$. By setting $T_0 > 0$ large enough, $[\omega_I(x) - T_0]^+ > 0$ if and only if $x \in [2^k - 1, 2^k)$ for some $k \in \mathbb{N}$, and in this case, $[\omega_I(x) - T_0]^+ \sim Cx/(\ln x)^{1+q}$. Then $K(x, T_0) \sim Cx/(\ln x)^q$. Because $\beta(x) \sim (\ln x)^2$, it is easy to check that $\tilde{u}(x)$ converges as $x \to \infty$. Then by Corollary 2.2, the SRT holds for U.

On the other hand, if $p^+ \in (0, 1)$, our argument for Theorem 2.1 requires L grow faster than $\ln x$. Meanwhile, it is desirable to have faster growth of L in order to get weaker conditions on $K(x, T_0)$. We have the following prior lower bound on the growth of L.

PROPOSITION 2.3. *Let (1.2) and (1.4) hold with $\alpha \in (0, 1)$. If for some $\kappa \in [0, 2\alpha)$,*

$$\omega_I(x) = O(x^\kappa), \tag{2.8}$$

then for any $\varepsilon \in (0, 2\alpha - \kappa)$, (2.2) holds with $L(x) = x^{\varepsilon/2}$.

Now we consider the SRT for the ladder height processes of S_n, with $S_0 = 0$. The (strict) ascending ladder height process H_n is defined to be S_{T_n}, where $T_0 = 0$, $T_n = \min\{k : S_k > H_{n-1}\}$, $n \geq 1$. The weak ascending ladder height process is defined by replacing $>$ with $\geq$ in the definition of T_n, and the descending process is defined by symmetry. Then H_n is a random walk such that the steps are i.i.d. $\sim H_1$. Denote $H = H_1$ and U^+ the renewal measure for H_n. As noted in the Introduction, in the next statement, we explicitly require $\alpha\varrho \in (0, 1/2]$.

THEOREM 2.4. *Let $\alpha \in (0,1)$ and (1.2)–(1.3) hold. Let $\alpha\varrho \in (0,1/2]$ and either* (a) $\varrho \in (0,1)$, *or* (b) $\varrho = 1$ *and $S_n \to \infty$ a.s. If there exist $T > 0$ and $c \in [0,\varrho)$, such that*

$$K(x,T) = o\big(A(x)^{2c}\big), \tag{2.9}$$

then the SRT holds for U^+,

$$\lim_{x\to\infty} x\mathbb{P}\{H > x\}U^+(x+I) = h\sin(\pi\alpha\varrho)/\pi,$$

where $h > 0$ is arbitrary if F is nonarithmetic, and is the span of F otherwise.

REMARK. Under the same conditions, the SRT also holds for the weak ladder process.

Now suppose X is infinitely divisible with Lévy measure ν, such that

$$\mathbb{E}[e^{\mathrm{i}\theta X}] = \exp\left\{\mathrm{i}\mu\theta - \sigma^2\theta^2/2 + \int (e^{\mathrm{i}\theta u} - 1 - \mathrm{i}\theta u\mathbf{1}\{|u| \le 1\})\nu(\mathrm{d}u)\right\}.$$

For $x > 0$, denote $\overline{\nu}(x) = \nu((x,\infty))$ and $\nu(-x) = \nu((-\infty,-x))$. Define

$$\tilde{K}(x,T) = \int_0^x [\tilde{\omega}_I(y) - T]^+ \,\mathrm{d}y \qquad \text{where } \tilde{\omega}_I(x) = x\nu(x+I)/\overline{\nu}(x).$$

THEOREM 2.5. *Let*

$$\overline{\nu}(x) \sim 1/A(x),\ \nu(-x) \sim r_\nu/A(x), \qquad x \to \infty, \tag{2.10}$$

where $A \in \mathcal{R}_\alpha$ with $\alpha \in (0,1/2]$ and $0 \le r_\nu < \infty$, and for some $\kappa \in [0,2\alpha)$,

$$\nu(x+I) = O\big(\overline{\nu}(x)/x^{1-\kappa}\big), \qquad x \to \infty. \tag{2.11}$$

Suppose Assumptions 1(b) *and* (c) *hold with $K(x,T_0)$ being replaced with $\tilde{K}(x,T_0)$ and $L(x) = x^{\varepsilon/2}$, where $\varepsilon \in (0,2\alpha-\kappa)$ is a fixed number. Then the SRT* (2.5) *holds for U, where $h > 0$ is arbitrary if ν is nonarithmetic, and is the span of ν otherwise.*

REMARK.

(1) Since (2.10) implies $\overline{F}(x) \sim 1/A(x)$ and $F(-x) \sim r_\nu/A(x)$ as $x \to \infty$ ([1], Theorem 8.2.1), as in Theorem 2.1, once Lemma 3.3 is established, the rest of the proof of Theorem 2.5 is standard.

(2) Condition (2.11) can be written as $x\nu(x+I)/\overline{\nu}(x) = O(x^\kappa)$. Therefore, it is analogous to (2.8) in Proposition 2.3. Indeed, our proof of Theorem 2.5 will rely on Proposition 2.3.

3. Proofs for SRT. We shall always denote $M_n = \max_{1\le i\le n} X_i$, and $J = (-h,h]$. Note $I - I = (-h,h) \subset J$.

3.1. *An auxiliary result.* Some of the notation and arguments in this subsection will also be used in the proof of Proposition 2.3. First, observe that since for any $x > 0$ and y, there are at most two $x + kh + J$, $k \in \mathbb{Z}$, that contain y, then for any $n \geq 0$ and event E,

$$\sum_k \mathbb{P}\{S_n \in x + kh + J, E\} = \mathbb{E}\left[\sum_k \mathbf{1}\{S_n \in x + kh + J\} I_E\right] \tag{3.1}$$
$$\leq \mathbb{E}(2I_E) = 2\mathbb{P}(E).$$

Let $Y_1, Y_2, \ldots$ be i.i.d. following the distribution of X conditional on $X > 0$ and denote

$$S_n^{\pm} = \sum_{i=1}^n X_i^{\pm}, \qquad N_n = \sum_{i=1}^n \mathbf{1}\{X_i > 0\}, \qquad V_n = \sum_{i=1}^n Y_i, \qquad \tilde{M}_n = \max_{i \leq n} Y_i.$$

Then $S_n = S_n^+ - S_n^-$ and

$$\mathbb{P}\{Y_i > x\} = \overline{F}(x^+)/p^+ \sim 1/\tilde{A}(x) \qquad \text{with } \tilde{A}(x) = p^+ A(x).$$

For $n \geq 1$ and $x > 0$, define

$$\zeta_{n,x} = a_n^{1-\gamma} x^{\gamma} \qquad \text{where } (1+\alpha)/(1+2\alpha) < \gamma < 1 \tag{3.2}$$

and

$$E_{n,x}^{(3)} = \{S_n^+ \in x + I, X_i > \zeta_{n,x} \text{ for at least two } i = 1, \ldots, n\},$$
$$E_{n,x}^{(2)} = \{S_n^+ \in x + I, M_n > x/2\} \setminus E_{n,x}^{(3)},$$
$$E_{n,x}^{(1)} = \{S_n^+ \in x + I, \zeta_{n,x} < M_n \leq x/2\} \setminus E_{n,x}^{(3)},$$
$$E_{n,x}^{(0)} = \{S_n^+ \in x + I\} \Big\backslash \bigcup_{i=1}^3 E_{n,x}^{(i)}.$$

Let $C_F, C'_F, \ldots$ denote constants that only depend on F (and possibly the fixed h) and may change from line to line. The auxiliary result we need is the following:

LEMMA 3.1. *Let* (1.2) *and* (1.4) *hold. Fix* $\delta \in (0, 1)$ *such that* $\delta^{\gamma\alpha/2} < p^+$ *and* $\delta^{1-\gamma} < 1/2$. *Let* $p = 1$ *if* $p^+ = 1$, *or* $p = 9p^+/10$ *if* $p^+ \in (0, 1)$. *Then for all* $x \gg 1$, $n_0 \gg 1$, $n_0 \leq n \leq A(\delta x)$, $pn \leq m \leq n$ *and* $T \geq 1$,

$$\mathbb{P}\{E_{n,x}^{(i)} | N_n = m\} \leq \frac{C_F n \overline{F}(x)}{x}\left[T + \frac{K(2x, T/3)}{a_n}\right], \qquad i = 3, 2 \tag{3.3}$$

and

$$\mathbb{P}\{E_{n,x}^{(i)} | N_n = m\} \leq \frac{C_F T n \overline{F}(x)}{x}, \qquad i = 1, 0. \tag{3.4}$$

PROOF. Notice that for $n \le A(\delta x)$, $\zeta_{n,x} \le (\delta x)^{1-\gamma} x^\gamma < x/2$ and $n \le \zeta_{n,x}$. Conditional on $N_n = m$, $S_n^+ \sim V_m$. Then for $1 \le m \le n$,

$$
\begin{aligned}
&\mathbb{P}\{E_{n,x}^{(3)} | N_n = m\} \\
&\quad \le m^2 \mathbb{P}\{V_m \in x + I, Y_{m-1} > \zeta_{n,x}, Y_m > \zeta_{n,x}\} \\
&\quad \le n^2 \sum_{k=0}^{\infty} \mathbb{P}\{V_m \in x + I, Y_{m-1} > \zeta_{n,x}, Y_m \in \zeta_{n,x} + kh + I\} \\
&\quad \le n^2 \sum_{k=0}^{\infty} \mathbb{P}\{V_{m-1} \in x - \zeta_{n,x} - kh + J, Y_{m-1} > \zeta_{n,x}, Y_m \in \zeta_{n,x} + kh + I\},
\end{aligned}
$$

where the last line is due to $I - I \subset J$. Then by independence of Y_i, the last inequality yields

$$
\mathbb{P}\{E_{n,x}^{(3)} | N_n = m\} \le \frac{n^2}{p^+} \sum_{k=0}^{\infty} \mathbb{P}\{X \in \zeta_{n,x} + kh + I\} Q_k, \tag{3.5}
$$

where

$$
Q_k = Q_k(m, n, x) = \mathbb{P}\{V_{m-1} \in x - \zeta_{n,x} - kh + J, Y_{m-1} > \zeta_{n,x}\}.
$$

To bound the RHS of (3.5), let

$$
D_k = D_k(n, x, T) = [\omega_I(\zeta_{n,x} + kh) - T]^+.
$$

Then for $k \ge 0$,

$$
\begin{aligned}
\mathbb{P}\{X \in \zeta_{n,x} + kh + I\} &\le \frac{\overline{F}(\zeta_{n,x} + kh)}{\zeta_{n,x} + kh}(T + D_k) \\
&\le \frac{\overline{F}(\zeta_{n,x})}{\zeta_{n,x}}(T + D_k).
\end{aligned}
$$

Next, for $x \gg 1$, $\zeta_{n,x} \ge \zeta_{1,x} > h$. Then for $k \ge x/h$, $x - \zeta_{n,x} - kh + h < x - kh < 0$, implying $Q_k(x) = 0$. Meanwhile, by (3.1),

$$
\sum_{k=0}^{\infty} Q_k \le 2\mathbb{P}\{Y_{m-1} > \zeta_{n,x}\} = \frac{2\overline{F}(\zeta_{n,x})}{p^+}.
$$

Combining (3.5) and the above bounds,

$$
\begin{aligned}
\mathbb{P}\{E_{n,x}^{(3)} | N_n = m\} &\le \frac{n^2 \overline{F}(\zeta_{n,x})}{p^+ \zeta_{n,x}} \sum_{k=0}^{\infty} (T + D_k) Q_k \\
&\le \frac{2Tn^2 \overline{F}(\zeta_{n,x})^2}{(p^+)^2 \zeta_{n,x}} + \frac{n^2 \overline{F}(\zeta_{n,x})}{p^+ \zeta_{n,x}} \sum_{0 \le k < x/h} Q_k D_k.
\end{aligned} \tag{3.6}
$$

For each k,

$$Q_k = \int_{(\zeta_{n,x},\infty)} \mathbb{P}\{V_{m-2} \in x - \zeta_{n,x} - kh - z + J\}\mathbb{P}\{X \in \mathrm{d}z | X > 0\} \tag{3.7}$$
$$\le \frac{\overline{F}(\zeta_{n,x})}{p^+} \sup_t \mathbb{P}\{V_{m-2} \in t + J\}.$$

By the LLTs ([1], Theorem 8.4.1–2) and the boundedness of the density g, for all $n \gg 1$ and $pn \le m \le n$, $\sup_t \mathbb{P}\{V_{m-2} \in t + J\} \le C_F/\tilde{A}(m) \le C'_F/a_n$. Consequently, by (3.7)

$$\sum_{0\le k<x/h} Q_k D_k \le \frac{C_F \overline{F}(\zeta_{n,x})}{a_n} \sum_{0\le k<x/h} D_k.$$

Then by (3.6),

$$\mathbb{P}\{E^{(3)}_{n,x} | N_n = m\} \le \frac{2Tn^2\overline{F}(\zeta_{n,x})^2}{(p^+)^2\zeta_{n,x}} + \frac{C_F n^2 \overline{F}(\zeta_{n,x})^2}{a_n p^+ \zeta_{n,x}} \sum_{0\le k<x/h} D_k. \tag{3.8}$$

Observe that $D_k = [\omega_I(\zeta_{n,x} + kh) - T]^+ \le [\omega_J(y) - T]^+$ for $y \in \zeta_{n,x} + kh + I$. Then for $x \gg 1$,

$$\sum_{0\le k<x/h} D_k \le \frac{1}{h} \sum_{0\le k<x/h} \int_{\zeta_{n,x}+kh}^{\zeta_{n,x}+kh+h} [\omega_J(y) - T]^+ \,\mathrm{d}y$$
$$\le \frac{1}{h} \int_0^{2x} [\omega_J(y) - T]^+ \,\mathrm{d}y.$$

Set $y_0 > 0$, such that for $y \ge y_0$,

$$\omega_J(y) = \frac{y\mathbb{P}\{X \in y - h + I\}}{\overline{F}(y)} + \frac{y\mathbb{P}\{X \in y + I\}}{\overline{F}(y)}$$
$$\le 2\omega_I(y - h) + \omega_I(y).$$

On $[0, y_0]$, $\omega_J(y) \le y_0/\overline{F}(y_0)$. By $[2\omega_I(y - h) + \omega_I(y) - T]^+ \le 2[\omega_I(y - h) - T/3]^+ + [\omega_I(y) - T/3]^+$,

$$\sum_{0\le k<x/h} D_k \le \frac{1}{h} \int_0^{y_0} \omega_J(y)\,\mathrm{d}y + \frac{1}{h} \int_{y_0}^{2x} [\omega_J(y) - T]^+ \,\mathrm{d}y \tag{3.9}$$
$$\le \frac{C'_F + C_F K(2x, T/3)}{h}.$$

This combined with (3.8) yields for all $T \ge 1$,

$$\mathbb{P}\{E^{(3)}_{n,x} | N_n = m\} \le \frac{C_F n^2 \overline{F}(\zeta_{n,x})^2}{\zeta_{n,x}} \left[T + \frac{K(2x, T/3)}{a_n}\right].$$

By [4], page 462, for $x \gg 1$ and $n \le A(x)$, $n\overline{F}(\zeta_{n,x})^2/\zeta_{n,x} \le C_F \overline{F}(x)/x$. Insert this inequality into the above one. Then (3.3) follows for $E^{(3)}_{n,x}$.

Since $E^{(2)}_{n,x} = \{S^+_n \in x + I$, one $X^+_i > x/2$, all other $X^+_i \le \zeta_{n,x}\}$, then

$$\begin{aligned}
&\mathbb{P}\{E^{(2)}_{n,x} | N_n = m\} \\
&\quad \le m\mathbb{P}\{V_m \in x + I, \tilde{M}_{m-1} \le \zeta_{n,x}, Y_m > x/2\} \\
&\quad \le m \sum_{k=0}^{\infty} \mathbb{P}\{V_m \in x + I, \tilde{M}_{m-1} \le \zeta_{n,x}, Y_m \in x/2 + kh + I\} \\
&\quad \le n \sum_{k=0}^{\infty} \mathbb{P}\{V_{m-1} \in x/2 - kh + J, \tilde{M}_{m-1} \le \zeta_{n,x}, Y_m \in x/2 + kh + I\}.
\end{aligned}$$

Denote $Q'_k = Q'_k(m, n, x) = \mathbb{P}\{V_{m-1} \in x/2 - kh + J, \tilde{M}_{m-1} \le \zeta_{n,x}\}$ and

$$D'_k = D'_k(n, x, T) = \bigl[\omega_I(x/2 + kh) - T\bigr]^+.$$

Then as the argument for (3.6),

$$\begin{aligned}
\mathbb{P}\{E^{(2)}_{n,x} | N_n = m\} &\le \frac{C_F n \overline{F}(x)}{x} \sum_{k=0}^{\infty} (T + D'_k) Q'_k \\
&\le \frac{C_F n \overline{F}(x)}{x} \biggl(2T + \sum_{0 \le k < x/h} Q'_k D'_k\biggr).
\end{aligned}$$

By the LLTs, $Q'_k \le \sup_t \mathbb{P}\{V_{m-1} \in t + J\} \le C_F/\tilde{A}(m) \le C'_F/a_n$. On the other hand, $\sum_{0 \le k < x/h} D'_k$ has the same bound (3.9). Then (3.3) follows for $E^{(2)}_{n,x}$.

To finish the proof, we need the next general result, which is essentially due to [3]; see also [4, 10] for results restricted to the arithmetic or operator cases.

LEMMA 3.2 (Denisov, Dieker and Shneer [3]). *Let* (1.2) *and* (1.4) *hold. There are* $C_F > 0$ *and* $C'_F > 0$, *such that for any positive sequence* $s_n \to \infty$,

$$\mathbb{P}\{S_n \in x + I, M_n \le s_n\} \le C'_F (1/s_n + 1/a_n) e^{-x/s_n + C_F n/A(s_n)}, \qquad \text{all } x > 0 \text{ and } n \gg 1.$$

Continuing the proof of Lemma 3.1, since $E^{(1)}_{n,x} = \{S^+_n \in x + I$, one $X^+_i \in (\zeta_{n,x}, x/2]$, all other $X^+_i \le \zeta_{n,x}\}$, then

$$\begin{aligned}
&\mathbb{P}\{E^{(1)}_{n,x} | N_n = m\} \\
&\quad \le m\mathbb{P}\{V_m \in x + I, \tilde{M}_{m-1} \le \zeta_{n,x} < Y_m \le x/2\} \\
&\quad = m \int_{(\zeta_{n,x}, x/2]} \mathbb{P}\{V_{m-1} \in x - z + I, \tilde{M}_{m-1} \le \zeta_{n,x}\}
\end{aligned}$$

$$\times \mathbb{P}\{X \in \mathrm{d}z | X > 0\}$$
$$\le \frac{n\overline{F}(\zeta_{n,x})}{p^+} \sup_{t \ge x/2} \mathbb{P}\{V_{m-1} \in t + I, \tilde{M}_{m-1} \le \zeta_{n,x}\}.$$

Since $pn \le m \le n \le A(\zeta_{n,x}) \le \tilde{A}(\zeta_{n,x})$, applying Lemma 3.2 to $\mathbb{P}\{V_{m-1} \in t + I, \tilde{M}_{m-1} \le \zeta_{n,x}\}$ for $t \ge x/2$, with $s_n = \zeta_{n,x}$,

$$\mathbb{P}\{E^{(1)}_{n,x} | N_n = m\} \le C_F n\overline{F}(\zeta_{n,x}) e^{-x/(2\zeta_{n,x})}/a_n, \qquad pn \le m \le n.$$

By $n\overline{F}(\zeta_{n,x}) \sim A(a_n)/A(\zeta_{n,x}) \le 1$ and $e^{-x/(2\zeta_{n,x})}/a_n \le C_F n\overline{F}(x)/x$ (cf. [4, 10]), (3.4) follows for $E^{(1)}_{n,x}$. Finally, by Lemma 3.2, for $pn \le m \le n$,

$$\mathbb{P}\{E^{(0)}_{n,x} | N_n = m\} = \mathbb{P}\{V_m \in x + I, \tilde{M}_m \le \zeta_{n,x}\} \le C_F e^{-x/\zeta_{n,x}}/a_n,$$

and (3.4) follows for $E^{(0)}_{n,x}$. □

PROOF OF LEMMA 3.2. This essentially is Lemma 7.1(iv) combined with Proposition 7.1 in [3]. That lemma assumes s_n to be some specific sequence and $F(-x)$ to be regularly varying at ∞. Both assumptions can be removed. To start with, for any distribution F and $s > 0$ with $F(s) > 0$, define $\tilde{F}(\mathrm{d}x) = e^{-\psi(s)+x/s}\mathbf{1}\{x \le s\}F(\mathrm{d}x)$, where

$$\psi(s) = \ln \mathbb{E}[e^{X/s}\mathbf{1}\{X \le s\}] \qquad \text{with } X \sim F.$$

Let $S_n = X_1 + \cdots + X_n$ with X_i i.i.d. $\sim F$ and $\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n$ with $\tilde{X}_i$ i.i.d. $\sim \tilde{F}$. Then

$$\mathbb{P}\{S_n \in x + I, M_n \le s\} \le e^{-x/s + n\psi(s)} \mathbb{P}\{\tilde{S}_n \in x + I\}.$$

By $\ln \mathbb{E}Z = \ln[1 + \mathbb{E}(Z-1)] \le \mathbb{E}(Z-1)$ for any $Z \ge 0$ and $e^x - 1 \le 2x$ for $x \le 1$, the following bounds hold:

$$\begin{aligned}\psi(s) &\le \mathbb{E}[e^{X/s}\mathbf{1}\{X \le s\} - 1] \\ &\le \mathbb{E}[(e^{X/s} - 1)\mathbf{1}\{X \le s\}] \\ &\le \mathbb{E}[(e^{X/s} - 1)\mathbf{1}\{0 < X \le s\}] \\ &\le 2s^{-1}\mathbb{E}[X\mathbf{1}\{0 < X \le s\}].\end{aligned}$$

By integration by parts and Karamata's theorem ([1], Theorem 1.5.11), (1.2) alone implies that for $p \ge 1$,

$$\begin{aligned}\int_0^s u^p F(\mathrm{d}u) &= p\int_0^s \overline{F}(u)u^{p-1}\,\mathrm{d}u - \overline{F}(s)s^p \\ &\sim \frac{\alpha s^p}{(p-\alpha)A(s)} \to \infty, \qquad s \to \infty\end{aligned} \tag{3.10}$$

and hence $\psi(s) \le 2s^{-1}\mathbb{E}[X\mathbf{1}\{0 < X \le s\}] \sim C_F/A(s)$. Let $\tilde{S}_n$ be defined with $s = s_n$. Then $\mathbb{P}\{S_n \in x + I, M_n \le s_n\} \le e^{-x/s_n + Cn/A(s_n)}\mathbb{P}\{\tilde{S}_n \in x + I\}$, and so it only remains to check

$$\mathbb{P}\{\tilde{S}_n \in x + I\} \le C_F(1/s_n + 1/a_n). \tag{3.11}$$

Since (1.4) holds as well, there is $s_0 > 0$, such that for $s > s_0$,

$$\begin{aligned} \int_{-s}^{0} |u|^p F(\mathrm{d}u) &\le p \int_0^s F(-u)u^{p-1}\,\mathrm{d}u \\ &\le C_F + rp\int_{s_0}^{s} \overline{F}(u)u^{p-1}\,\mathrm{d}u \\ &\le C_F s^p/A(s). \end{aligned} \tag{3.12}$$

Let $\mu_p(s) := \mathbb{E}[|X|^p\mathbf{1}\{|X| \le s\}]$. Then for $s \gg 1$, by (3.10) and (3.12),

$$C_F s^p/A(s) \le \mathbb{E}\big[X^p\mathbf{1}\{0 < X \le s\}\big] \le \mu_p(s) \le C'_F s^p/A(s). \tag{3.13}$$

It follows that $\limsup_{x\to\infty} x^2\overline{G}(x)/\mu_2(x) < \infty$, where $G(x) = \mathbb{P}\{|X| \le x\}$, so by Proposition 7.1 in [3], for all $n \gg 1$,

$$\sup_x \mathbb{P}\{\tilde{S}_n \in x + I\} \le C_F(1/s_n + 1/r_n), \tag{3.14}$$

where $r_n > 0$ is the solution to $Q(x) := x^{-2}\mu_2(x) + \overline{G}(x) = 1/n$, which exists and is unique for all $n \gg 1$. On the one hand, since $Q(x) \ge \overline{F}(x) \sim 1/A(x)$, $r_n \ge C_F a_n$. On the other, by (3.13), $Q(x) \le C_F/A(x)$ and then $r_n \le C'_F a_n$. Then (3.11) follows from (3.14). □

3.2. *Proof of Theorem* 2.1. We need two lemmas for the proof of Theorem 2.1.

LEMMA 3.3. *Let* (1.2) *and* (1.4) *hold. Then Assumption* 1 *implies*

$$\lim_{\delta\to 0+}\limsup_{x\to\infty} \frac{x}{A(x)} \sum_{n\le A(\delta x)} \mathbb{P}\{S_n \in x + I\} = 0.$$

LEMMA 3.4. *Let* (1.2) *and* (1.3) *hold. Given* $0 < \delta < 1$, *let* $J_\delta(x) = (A(\delta x), A(x/\delta))$. *Then*

$$\lim_{x\to\infty} \frac{x}{A(x)} \sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in x + I\} = \alpha h \int_\delta^{1/\delta} x^{-\alpha} g(x)\,\mathrm{d}x. \tag{3.15}$$

Assume the lemmas are true for now. Since $\mathbb{P}\{S_n \in x + I\} = O(1/a_n)$ and

$$\sum_{n\ge A(x/\delta)} 1/a_n \sim \frac{A(x/\delta)}{(\alpha^{-1}-1)x/\delta} \sim \delta^{1-\alpha}\frac{A(x)}{(\alpha^{-1}-1)x},$$

by (3.15),

$$\lim_{x\to\infty}\frac{x}{A(x)}\sum_{n>A(\delta x)}\mathbb{P}\{S_n\in x+I\}=\alpha h\int_0^{1/\delta}x^{-\alpha}g(x)\,\mathrm{d}x.$$

Combining this with Lemma 3.3 and letting $\delta\to 0+$, we then get (2.5).

PROOF OF LEMMA 3.3. Denote $\Omega_{n,x}=\{S_n\in x+I\}$. By Assumption 1, it suffices to show

$$\lim_{\delta\to 0+}\limsup_{x\to\infty}\frac{x}{A(x)}\sum_{L(x)\le n\le A(\delta x)}\mathbb{P}(\Omega_{n,x})=0. \tag{3.16}$$

Let $\delta>0$ such that $\delta^{-\gamma\alpha/2}p^+>1>2\delta^{1-\gamma}$. Set $p=1$ if $p^+=1$, and $p=9p^+/10$ if $p^+<1$. Then

$$\mathbb{P}(\Omega_{n,x})\le\sum_{pn\le m\le n}\mathbb{P}(\Omega_{n,x}|N_n=m)\mathbb{P}\{N_n=m\}+\mathbb{P}\{N_n<pn\}.$$

For each $pn\le m\le n$, conditional on $N_n=m$, S_n^+ and S_n^- are independent. Therefore,

$$\begin{aligned}&\mathbb{P}(\Omega_{n,x}|N_n=m)\\&\quad=\int_0^\infty\mathbb{P}\{S_n^+\in x+z+I|N_n=m\}\mathbb{P}\{S_n^-\in\mathrm{d}z|N_n=m\}.\end{aligned}$$

Since $\{S_n^+\in x+z+I\}=\bigcup_{i=0}^4 E_{n,x+z}^{(i)}$, by Lemma 3.1, for $x\gg 1$, $L(x)\le n\le A(\delta x)$ and $pn\le m\le n$,

$$\begin{aligned}&\mathbb{P}(\Omega_{n,x}|N_n=m)\\&\quad\le C_F n\int_0^\infty\frac{\overline{F}(x+z)}{x+z}\left[T_0+\frac{K(2x+2z,T_0)}{a_n}\right]\mathbb{P}\{S_n^-\in\mathrm{d}z|N_n=m\}\\&\quad=C_F n\mathbb{E}\left\{\frac{\overline{F}(x_n)}{x_n}\left[T_0+\frac{K(2x_n,T_0)}{a_n}\right]\middle|N_n=m\right\},\end{aligned}$$

where $x_n=x+S_n^-$. N_n is the sum of n independent Bernoulli random variables each with mean p^+. If $p^+\in(0,1)$, then by Chernoff's inequality, for $n\gg 1$, $\mathbb{P}\{N_n<pn\}\le e^{-\lambda n}$, where $\lambda=\lambda(p^+)>0$ is a constant; cf. [17], Corollary 1.9. As a result,

$$\mathbb{P}(\Omega_{n,x})\le C_F n\mathbb{E}\left\{\frac{\overline{F}(x_n)}{x_n}\left[T_0+\frac{K(2x_n,T_0)}{a_n}\right]\right\}+e^{-\lambda n}. \tag{3.17}$$

If $p^+=1$, then $N_n\equiv n$ and $S_n^-=0$, so by setting $\lambda=\infty$, the above inequality still holds.

Since $x_n \ge x$, given $c > 1$, $\overline{F}(x_n)/x_n \le \overline{F}(x)/x \le c/[xA(x)]$ for $x \gg 1$. Then, writing

$$R(x) = \frac{x}{A(x)} \sum_{L(x)\le n\le A(\delta x)} \frac{n}{a_n} \mathbb{E}\left[\frac{K(2x_n, T_0)}{x_n A(x_n)}\right],$$

we have

$$\begin{aligned}\frac{x}{2A(x)} &\sum_{L(x)\le n\le A(\delta x)} n\mathbb{E}\left\{\frac{\overline{F}(x_n)}{x_n}\left[T_0 + \frac{K(2x_n, T_0)}{a_n}\right]\right\} \\ &\le \frac{T_0}{A(x)^2} \sum_{L(x)\le n\le A(\delta x)} n + R(x).\end{aligned}$$

The first term on the RHS is $O(A(\delta x)^2/A(x)^2) = \delta^{2\alpha}$. To bound $R(x)$, write $\theta = 1/\alpha - 1$. Then $\theta > 0$ and $n/a_n = n^{-\theta}/\beta(n)$. Consider two cases.

Case 1: $\alpha \in (0, 1/2)$. Then $\theta > 1$. By Assumption 1,

$$K(2x_n, T_0) = o\big(A(2x_n)^2/u_\theta(2x_n)\big),$$

and since $L \in \mathcal{R}_c$ with $c \in [0, \alpha]$, $u_\theta \in \mathcal{R}_{c(1-\theta)}$. As a result,

$$\begin{aligned} R(x) &\le \frac{x}{A(x)}\left(\sum_{n\ge L(x)} \frac{n}{a_n}\right) \max_{n\ge L(x)} \mathbb{E}\left[\frac{K(2x_n, T_0)}{x_n A(x_n)}\right] \\ &= o\left(\frac{x u_\theta(x)}{A(x)} \max_{n\ge L(x)} \mathbb{E}\left[\frac{A(x_n)}{x_n u_\theta(x_n)}\right]\right), \qquad x \to \infty.\end{aligned}$$

Since $A(x)/xu_\theta(x) \in \mathcal{R}_b$ with

$$b = \alpha - 1 + c(\theta - 1) \le \alpha - 1 + \alpha(1/\alpha - 2) = -\alpha < 0,$$

then $\mathbb{E}[A(x_n)/x_n u_\theta(x_n)] = O(A(x)/xu_\theta(x))$. It follows that $R(x) \to 0$ as $x \to \infty$.

Case 2: $\alpha = 1/2$. Since $x/A(x) \le 2x_n/A(x_n)$ for $x \gg 1$,

$$R(x) = O(1) \sum_{L(x)\le n\le A(\delta x)} \frac{n}{a_n} \mathbb{E}\left[\frac{K(2x_n, T_0)}{A(x_n)^2}\right].$$

If $\tilde{u}(x)/\tilde{u}(L(x)) \to 1$, then by Assumption 1,

$$K(2x_n, T_0)/A(x_n)^2 = O\big(1/\tilde{u}(2x_n)\big).$$

Since $\tilde{u}(x)$ is increasing, then

$$\begin{aligned} R(x) &= O\big(1/\tilde{u}(x)\big) \sum_{A(L(x))\le n\le A(\delta x)} \frac{n^{-1}}{\beta(n)} \\ &= O\big(1/\tilde{u}(x)\big) \int_{A(L(x))}^{A(x)} \frac{y^{-1}}{\beta(y)}\,\mathrm{d}y = o(1).\end{aligned}$$

If $\tilde{u}(x)/\tilde{u}(L(x)) \not\to 1$, then by Assumption 1,

$$K(2x_n, T_0)/A(x_n)^2 = o\big(1/\tilde{u}(2x_n)\big) = o\big(1/\tilde{u}(x)\big),$$

and hence

$$R(x) = o\big(1/\tilde{u}(x)\big) \sum_{n \le A(\delta x)} \frac{n^{-1}}{\beta(n)} = o(1).$$

Thus, for all $\alpha \in (0, 1/2]$, $R(x) = o(1)$. Finally, if $p^+ \in (0, 1)$, then given $c > 0$ such that $\lambda c > 1 - \alpha$, for $x \gg 1$, $\sum_{n \ge L(x)} e^{-\lambda n} \le \sum_{n \ge c \ln x} e^{-\lambda n} = O(x^{-\lambda c}) = o(A(x)/x)$. Then by summing (3.17) over $L(x) \le n \le A(\delta x)$ and taking the limit as $x \to \infty$ followed by $\delta \to 0$, the proof is complete. □

PROOF OF LEMMA 3.4. If X is arithmetic or nonlattice, then (3.15) is well-known [4, 7, 9, 18]. The only remaining case is where X is lattice but nonarithmetic. While Theorem 2 in [7] correctly states that (3.15) still holds in this case, the argument therein cannot establish the fact as it overlooks issues caused by the discrete nature of X.

Let X be concentrated in $a + d\mathbb{Z}$ with $a/d > 0$ being irrational and $d > 0$ the span. If $h \ge d$, then choose $k \in \mathbb{N}$ such that $h' = h/k < d$. Letting $I' = (0, h']$ and $x_j = x + jh'$, $\mathbb{P}\{S_n \in x + I\} = \sum_{j=0}^{k-1} \mathbb{P}\{S_n \in x_j + I'\}$, $x_j/A(x_j) \sim x/A(x)$, and hence if (3.15) holds for $\mathbb{P}\{S_n \in x + I'\}$, it holds for $\mathbb{P}\{S_n \in x + I\}$ as well. Thus, without loss of generality, let $0 < h < d$.

For $z \in \mathbb{R}$, denote $L_z := d\mathbb{Z} \cap (z + I)$. Since $h < d$, L_z contains at most one point. For $x > 0$ and $n \ge 1$, if $L_{x-na} = \{dk\}$, by Gnedenko's LLT, $\mathbb{P}\{S_n \in x + I\} = \mathbb{P}\{S_n = na + dk\} = (d/a_n)[g((na + kd)/a_n) + o(1)]$ as $n \to \infty$, where $o(1)$ is uniform in x ([1], Theorem 8.4.1). Since g has bounded derivative and $|x - (na + kd)| < h$, it is seen

$$\mathbb{P}\{S_n \in x + I\} = \mathbf{1}\{L_{x-na} \ne \varnothing\}(d/a_n)\big[g(x/a_n) + o(1)\big].$$

For $x \gg 1$ and $n \in J_\delta(x)$, $x/a_n \in (\delta, 1/\delta)$. Since $g > 0$ on $[\delta, 1/\delta]$, the above display can be written as

$$\mathbb{P}\{S_n \in x + I\} = \mathbf{1}\{L_{x-na} \ne \varnothing\}(d/a_n)\big[1 + \varepsilon_n(x)\big]g(x/a_n), \tag{3.18}$$
$$\text{with } \lim_{x\to\infty} \sup_{n \in J_\delta(x)} \big|\varepsilon_n(x)\big| = 0.$$

Let $m = m(x)$ and $M = M(x)$ be the smallest and largest integers in $(A(\delta x), A(x/\delta) + 1)$, respectively. Fix integers $m = N_1 < N_2 < \cdots < N_s < N_{s+1} = M$, where $N_i = N_i(x)$ and $s = s(x)$, such that as $x \to \infty$,

$$\min_{1 \le i \le s} [N_{i+1} - N_i] \to \infty, \qquad \max_{1 \le i \le s} [a_{N_{i+1}} - a_{N_i}] = o(x).$$

Then by (3.18)

$$\sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in x+I\} \sim d \sum_{j=1}^{s} \sum_{n=N_j}^{N_{j+1}-1} \mathbf{1}\{L_{x-na} \neq \varnothing\} \frac{g(x/a_n)}{a_n}.$$

By the choice of $N_1, \dots, N_{s+1}$, for each $1 \le j \le s$ and $n = N_j, \dots, N_{j+1}-1$,

$$g(x/a_n)/a_n = \big[1+\varepsilon_n(x)\big] g(x/a_{N_j})/a_{N_j}, \quad \text{with } \lim_{x\to\infty} \sup_{n\in J_\delta(x)} \big|\varepsilon_n(x)\big| = 0. \tag{3.19}$$

Thus

$$\sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in x+I\} \sim d \sum_{j=1}^{s} \frac{g(x/a_{N_j})}{a_{N_j}} \sum_{n=N_j}^{N_{j+1}-1} \mathbf{1}\{L_{x-na} \neq \varnothing\}.$$

Denote $K = \{\omega \in \mathbb{C} : |\omega| = 1\}$. Then $L_z \neq \varnothing \iff e^{2\pi \mathrm{i} z/d}$ falls into the arc $\Gamma = \{\omega = e^{2\pi \mathrm{i}\theta} : \theta \in [-h/d, 0)\} \subset K$. Let $c = a/d$ and define $T : K \to K$ as $T(\omega) = \omega e^{-2\pi \mathrm{i} c}$. Let $\omega_j = e^{2\pi \mathrm{i}(x - N_j a)/d}$. Then

$$\sum_{n=N_j}^{N_{j+1}-1} \mathbf{1}\{L_{x-na} \neq \varnothing\} = \sum_{n=0}^{N_{j+1}-N_j-1} \mathbf{1}\big\{T^n(\omega_j) \in \Gamma\big\}.$$

Since c is irrational, T is a homeomorphism of K with no periodic points, that is, for any $\omega \in K$ and $n \in \mathbb{N}$, $T^n(\omega) \neq \omega$. Then by ergodic theory ([19], Section 6.5), for any $f \in C(K)$, $(1/N)\sum_{n=0}^{N-1} f(T^n\omega) \to \int f \,\mathrm{d}\mu$ uniformly in $\omega \in K$, with μ the uniform probability measure on K. Since $\mu(\Gamma) = h/d$, and for any $\varepsilon > 0$, there are f, $g \in C(K)$ with $0 \le f(\omega) \le \mathbf{1}\{\omega \in \Gamma\} \le g(\omega) \le 1$ such that $0 \le \int (g-f)\,\mathrm{d}\mu < \varepsilon$, then

$$\sum_{n=0}^{N_{j+1}-N_j-1} \mathbf{1}\big\{T^n(\omega_j) \in \Gamma\big\} = (N_{j+1}-N_j)\big[1+\varepsilon_j(x)\big](h/d),$$

$$\text{with } \lim_{x\to\infty} \sup_{1\le j\le s} \big|\varepsilon_j(x)\big| = 0.$$

This combined with the previous two displays and then with (3.19) yields

$$\begin{aligned}\sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in x+I\} &\sim d \sum_{j=1}^{s} \frac{g(x/a_{N_j})}{a_{N_j}} (N_{j+1}-N_j)(h/d) \\ &\sim h \sum_{j=1}^{s} \sum_{n=N_j}^{N_{j+1}-1} \frac{g(x/a_n)}{a_n} = h \sum_{n\in J_\delta(x)} \frac{g(x/a_n)}{a_n}.\end{aligned}$$

Multiply both sides by $x/A(x)$ and let $x \to \infty$. Standard derivation such as the one on page 366 in [1] then yields (3.15). □

3.3. *Proof of Proposition* 2.3. Given $0 < \varepsilon < 2\alpha - \kappa$, fix $c \in (\varepsilon/(2\alpha), 1)$ such that $1 + \varepsilon < c(\alpha + 1 - \kappa) + \alpha$. Since X_i are i.i.d.,

$$\begin{aligned}\mathbb{P}\{S_n \in x + I, M_n > x^c\} &\le n\mathbb{P}\{S_n \in x + I, X_n > x^c\} \\ &= n\sum_{k=0}^{\infty}\mathbb{P}\{S_n \in x + I, X_n \in x^c + kh + I\} \\ &\le n\sum_{k=0}^{\infty}\mathbb{P}\{S_{n-1} \in x - x^c - kh + J, X_n \in x^c + kh + I\} \\ &= n\sum_{k=0}^{\infty}\mathbb{P}\{S_{n-1} \in x - x^c - kh + J\}\mathbb{P}\{X \in x^c + kh + I\}.\end{aligned}$$

Then by (3.1), $\mathbb{P}\{S_n \in x + I, M_n > x^c\} \le 2n \sup_{t \ge x^c} \mathbb{P}\{X \in t + I\}$. By assumption, for all $t \ge x^c$, $\mathbb{P}\{X \in t + I\} \le C/[t^{1-\kappa}A(t)] \le C/[x^{c(1-\kappa)}A(x^c)]$, where $C > 0$ is a constant that may change from line to line. Then by the choice of c,

$$\begin{aligned}x\overline{F}(x)\sum_{n \le x^{\varepsilon/2}}\mathbb{P}\{S_n \in x + I, M_n > x^c\} &\le \frac{Cx}{A(x)}\sup_{t \ge x^c}\mathbb{P}\{X \in t + I\}\sum_{n \le x^{\varepsilon/2}} n \\ &\le \frac{Cx^{1+\varepsilon}}{x^{c(1-\kappa)}A(x^c)A(x)} = o(1), \qquad x \to \infty.\end{aligned} \tag{3.20}$$

Note that if $S_n \in x + I$ and $M_n \le x^c$, then $n \ge x^{1-c}$. By $x^{\varepsilon/2} = o(A(x^c))$ and Lemma 3.2,

$$\begin{aligned}\sum_{n \le x^{\varepsilon/2}}\mathbb{P}\{S_n \in x + I, M_n \le x^c\} &= \sum_{x^{1-c} \le n \le x^{\varepsilon/2}}\mathbb{P}\{S_n \in x + I, M_n \le x^c\} \\ &\le C\sum_{x^{1-c} \le n \le x^{\varepsilon/2}}(1/x^c + 1/a_n)e^{-x^{1-c}} \\ &\le o(e^{-x^{1-c}}).\end{aligned}$$

Then $x\overline{F}(x)\sum_{n \le x^{\varepsilon/2}}\mathbb{P}\{S_n \in x + I, M_n \le x^c\} = o(1)$, which together with (3.20) completes the proof.

3.4. *Proof of Theorem* 2.4. Since $\alpha \in (0, 1)$, it is known that $A^+ \in \mathcal{R}_{\alpha\varrho}$, that is, $A^+(x)$ is regularly varying with exponent $\alpha\varrho$ [14]. Let $\omega_I^+(x)$ and $K^+(x, T)$ denote the functions defined by (1.6) and (1.7) for H. By assumption, $K(x, T) = O(x^{2c\alpha})$ for some $c \in [0, \varrho)$ and $T > 0$. We shall show that for any $\gamma \in (c, \varrho)$,

$$K^+(x, T) = O(x^{2\gamma\alpha}). \tag{3.21}$$

Once this is proved, then the proof follows from Corollary 2.2. For $t > 0$,

$$\mathbb{P}\{H \in t + I\} = \int_0^\infty \mathbb{P}\{X \in t + y + I\} U^-(\mathrm{d}y),$$

where $U^-(\mathrm{d}t) = \sum_{n=0}^\infty \mathbb{P}\{H_n^- \in -\mathrm{d}t\}$ concentrates on $[0, \infty)$, with H_n^- the weak decreasing latter process of S_n ([8], page 399). Then

$$\begin{aligned}[\omega_I^+(t) - T]^+ &= \frac{1}{\mathbb{P}\{H > t\}}[t\mathbb{P}\{H \in t + I\} - \mathbb{P}\{H > t\}T]^+ \\ &\le \frac{1}{\mathbb{P}\{H > t\}} \int_0^\infty [t\mathbb{P}\{X \in t + y + I\} - \overline{F}(t + y)T]^+ U^-(\mathrm{d}y) \\ &\le \frac{1}{\mathbb{P}\{H > t\}} \int_0^\infty \frac{t\overline{F}(t + y)}{t + y}[\omega_I(t + y) - T]^+ U^-(\mathrm{d}y).\end{aligned}$$

Denote $g_y(t) = t\overline{F}(t + y)/(t + y)$. Then

$$K^+(x, T) \le \sum_{i=1}^4 I_i,$$

$$\text{with } I_i = \int_{A_i} \frac{g_y(t)[\omega_I(t + y) - T]^+}{\mathbb{P}\{H > t\}} \mathrm{d}t\, U^-(\mathrm{d}y), \tag{3.22}$$

where $A_1 = \{0 \le t \le x < y\}$, $A_2 = \{0 \le t < y \le x\}$, $A_3 = \{M \le y \le t \le x\}$, and $A_4 = \{0 < y < M, y \le t \le x\}$, where $M \gg 1$ is a fixed number. Fix $0 < \beta < \alpha$.

First, let $\varrho \in (0, 1)$. For $(t, y) \in A_1$, $\mathbb{P}\{H > t\} \ge \mathbb{P}\{H > x\}$. Let $x \gg 1$. Then $g_y(t) \le h_y(t) := t/(t + y)^{1+\beta}$ and

$$I_1 \le \frac{1}{\mathbb{P}\{H > x\}} \int_0^x \mathrm{d}t \int_x^\infty h_y(t)[\omega_I(t + y) - T]^+ U^-(\mathrm{d}y).$$

Since for each $y \ge x$, $h_y(t)$ is increasing on $[0, x]$,

$$\begin{aligned}I_1 &\le \frac{1}{\mathbb{P}\{H > x\}} \int_0^x \mathrm{d}t \int_x^\infty h_y(x)[\omega_I(t + y) - T]^+ U^-(\mathrm{d}y) \\ &\le \frac{1}{\mathbb{P}\{H > x\}} \int_x^\infty h_y(x) K(x + y, T) U^-(\mathrm{d}y) \\ &\le \frac{C}{\mathbb{P}\{H > x\}} \int_x^\infty \frac{x}{(x + y)^{1+\beta-2c\alpha}} U^-(\mathrm{d}y),\end{aligned}$$

where $C > 0$ is a constant. Since H^- is in the domain of attraction of stable law with exponent $\alpha(1 - \varrho)$ [6], $U^-(x\,\mathrm{d}u)/x^{\alpha(1-\varrho)}$ converges vaguely to $Cu^{\alpha(1-\varrho)-1}\mathbf{1}\{u > 0\}\,\mathrm{d}u$ as $x \to \infty$, where $C > 0$ is a constant; see [1], pages 361–363. Therefore, by variable substitute $y = xu$,

$$I_1 \le \frac{Cx^{-\beta+2c\alpha+\alpha(1-\varrho)}}{\mathbb{P}\{H > x\}} \int_1^\infty \frac{u^{\alpha(1-\varrho)-1}\,\mathrm{d}u}{(1 + u)^{1+\beta-2c\alpha}}.$$

As long as $0 < \alpha - \beta \ll 1$, the integral is finite. (Recall that $c\alpha < \alpha\varrho \le 1/2$.) Then, for any $\gamma > c$, $I_1 = O(x^{2\gamma\alpha})$.

To bound I_2, observe $\mathbb{P}\{H > t\} \ge \mathbb{P}\{H > y\}$ for $(t, y) \in A_2$. Then by $g_y(t) \le \overline{F}(y)$,

$$
\begin{aligned}
I_2 &\le \int_0^x \frac{U^-(\mathrm{d}y)}{\mathbb{P}\{H > y\}} \int_0^y g_y(t)\big[\omega_I(t+y) - T\big]^+ \,\mathrm{d}t \\
&\le \int_0^x \frac{\overline{F}(y)U^-(\mathrm{d}y)}{\mathbb{P}\{H > y\}} \int_0^y \big[\omega_I(t+y) - T\big]^+ \,\mathrm{d}t \\
&\le \int_0^x \frac{\overline{F}(y)K(2y, T)}{\mathbb{P}\{H > y\}} U^-(\mathrm{d}y).
\end{aligned}
$$

By assumption, $\overline{F}(y)K(2y, T)/\mathbb{P}\{H > y\} = O(y^{-\beta+2c\alpha+\alpha\varrho})$ for any $\beta < \alpha$. Since $U^-((0, x])$ is regularly varying with exponent $\alpha(1-\varrho)$, the integral is of order $O(x^{2\gamma\alpha})$ for any $\gamma > c$.

Let $M \gg 1$ such that $\overline{F}(t+y)/\mathbb{P}\{H > t\} < k_y(t) := t^{\alpha\varrho}/(t+y)^\beta$ for $(t, y) \in A_3$. If $\beta \in (\alpha\varrho, \alpha)$, then $k_y(t)$ has maximum value $C/y^{\beta-\alpha\varrho}$, where $C = C(\beta) > 0$ is a constant. Then

$$
\begin{aligned}
I_3 &\le \int_{A_3} \frac{Ct}{(t+y)y^{\beta-\alpha\varrho}} \big[\omega_I(t+y) - T\big]^+ \,\mathrm{d}t\, U^-(\mathrm{d}y) \\
&\le \int_M^x \frac{CU^-(\mathrm{d}y)}{y^{\beta-\alpha\varrho}} \int_y^x \big[\omega_I(t+y) - T\big]^+ \,\mathrm{d}t \\
&\le K(2x, T) \int_M^x \frac{CU^-(\mathrm{d}y)}{y^{\beta-\alpha\varrho}}.
\end{aligned}
$$

The integral is of order $O(x^{\alpha-\beta})$. Then by the assumption on K, $I_3 = O(x^{2\gamma\alpha})$ for any $\gamma > c$.

For I_4, since $g_y(t)/\mathbb{P}\{H > t\} \le \overline{F}(t)/\mathbb{P}\{H > t\}$ is bounded,

$$
\begin{aligned}
I_4 &\le \int_0^M U^-(\mathrm{d}y) \int_y^x \big[\omega_I(t+y) - T\big]^+ \,\mathrm{d}t \\
&\le K(2x, T)U^-\big([0, M)\big) = O\big(x^{2c\alpha}\big).
\end{aligned}
$$

Combining the above bounds for I_i and (3.22), then (3.21) follows when $\varrho \in (0, 1)$.

If $\varrho = 1$ and $S_n \to \infty$ a.s., then U^- is a finite measure ([8], pages 395–396). It is then not hard to see the above bounds for I_i still hold. The proof is then complete.

3.5. *Proof of Theorem* 2.5. From the proof of Theorem 2.1, it suffices to prove Lemma 3.3 under the assumptions on the Lévy measure ν of X. We will use several times the fact that $X \sim Y_1 + \cdots + Y_N + W$, where Y_i, N and W are independent,

$$
Y_i \sim Y \sim G(x) = \mathbf{1}\{x > 1\}\nu\big((1, x)\big)/\nu_0,
$$

with $\nu_0 = \overline{\nu}(1)$, $N \sim \text{Poisson}(\nu_0)$, and for $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\mathrm{i}\theta W}] = \exp\left\{\mathrm{i}\mu\theta - \sigma^2\theta^2/2 + \int \left(e^{\mathrm{i}\theta u} - 1 - \mathrm{i}\theta u \mathbf{1}\{|u| \le 1\}\right)\mathbf{1}\{u \le 1\}\nu(\mathrm{d}u)\right\}.$$

Then $\mathbb{E}[e^{tW}] < \infty$ for any $t > 0$ ([15], Theorem 25.17). Write $\zeta_N = Y_1 + \cdots + Y_N$, and when N is random, always assume that it is independent of Y_i.

LEMMA 3.5. *Let* (2.10) *and* (2.11) *hold. Then given* $c \in (0, 2\alpha - \kappa)$, (2.2) *holds with* $L(x) = x^{c/2}$.

By this lemma, it suffices to show

$$\lim_{\delta \to 0+} \limsup_{x \to \infty} \frac{x}{A(x)} \sum_{L(x) \le n \le A(\delta x)} \mathbb{P}(\Omega_{n,x}) = 0,$$

where $L(x) = x^{\varepsilon/2}$ and $\Omega_{n,x} = \{S_n \in x + I\}$. For $n \ge 1$,

$$S_n \sim \zeta_{N_n} + V_n \qquad \text{with } V_n = W_1 + \cdots + W_n,$$

where $N_n \sim \text{Poisson}(n\nu_0)$, and $W_i \sim W$ are independent random variables. Then

$$\mathbb{P}(\Omega_{n,x}) \le \int_{-\infty}^{x/2} \mathbb{P}\{\zeta_{N_n} \in x - z + I\}\mathbb{P}\{V_n \in \mathrm{d}z\} + \mathbb{P}\{V_n \ge x/2\}. \tag{3.23}$$

Since $\mu := \ln \mathbb{E}[e^W] < \infty$, $\mathbb{P}\{V_n \ge x/2\} \le \mathbb{E}[e^{V_n - x/2}] = e^{n\mu - x/2}$. Therefore,

$$\max_{n \le A(\delta x)} \mathbb{P}\{V_n \ge x/2\} \le \max_{n \le A(\delta x)} e^{n\mu - x/2} = O\left(e^{-x/4}\right). \tag{3.24}$$

Next, for $z \le x/2$,

$$\begin{aligned} &\mathbb{P}\{\zeta_{N_n} \in x - z + I\} \\ &\quad \le \sum_{k > n\nu_0/2} \mathbb{P}\{\zeta_k \in x - z + I\}\mathbb{P}\{N_n = k\} + \mathbb{P}\{N_n \le n\nu_0/2\}. \end{aligned} \tag{3.25}$$

Since $\mathbb{E}[e^{-N_n}] = e^{n\nu_0(1/e - 1)}$, by Markov's inequality,

$$\max_{n \ge L(x)} \mathbb{P}\{N_n \le n\nu_0/2\} \le \max_{n \ge L(x)} e^{-n\nu_0(1/2 - 1/e)} \le e^{-L(x)\nu_0/10}. \tag{3.26}$$

On the other hand, note that for $t > 1$, $\mathbb{P}\{Y \in t + I\} = \nu(t + I)/\nu_0$ and $\overline{G}(t) = \overline{\nu}(t)/\nu_0$. Then, as $x - z \ge x/2$ and $Y_i > 1$, for each $k > n\nu_0/2 \ge L(x)\nu_0/2$, by Lemma 3.1,

$$\mathbb{P}\{\zeta_k \in x - z + I\} \le \frac{C_\nu k \overline{\nu}(x - z)}{x - z}\left[T_0 + \frac{\tilde{K}(2(x - z), T_0)}{a_k}\right],$$

where C_ν is a constant only depending on ν. Then by (3.23)–(3.26), letting $x_n = x - V_n$,

$$\mathbb{P}(\Omega_{n,x}) \le C_\nu \mathbb{E}\left[\mathbf{1}\{V_n \le x/2\}\frac{N_n \overline{\nu}(x_n)}{x_n}\left[T_0 + \frac{\tilde{K}(2x_n, T_0)}{a_{N_n}}\right]\right] + \varepsilon_n(x),$$

where $\max_{L(x)\le n\le A(\delta x)} \varepsilon_n(x) = o(x^{-M})$ for any $M > 0$. Note that N_n and V_n are independent. Since $\overline{\nu}(x)/x$ is regularly varying and decreasing, then

$$\mathbb{P}(\Omega_{n,x}) \le C_\nu T_0 n \frac{\overline{\nu}(x)}{x} + C'_\nu \mathbb{E}\left[\frac{N_n}{a_{N_n}}\right]\mathbb{E}\left[\mathbf{1}\{x_n \ge x/2\}\frac{\overline{\nu}(x_n)\tilde{K}(2x_n, T_0)}{x_n}\right] + \varepsilon_n(x).$$

Since

$$\mathbb{E}\left[\frac{N_n}{a_{N_n}}\mathbf{1}\{N_n < n\nu_0/2 \text{ or } N_n > 2n\nu_0\}\right] = o(e^{-cn}), \qquad n \to \infty,$$

where $c > 0$ is a constant, then by dominated convergence,

$$\frac{a_n}{n}\mathbb{E}\left[\frac{N_n}{a_{N_n}}\right] \sim \mathbb{E}\left[\frac{N_n/n}{a_{N_n}/a_n}\mathbf{1}\{n\nu_0/2 \le N_n \le 2n\nu_0\}\right] \sim \nu_0^{1-1/\alpha}.$$

Consequently,

$$\mathbb{P}(\Omega_{n,x}) \le C_\nu T_0 n \frac{\overline{\nu}(x)}{x} + C_\nu \frac{n}{a_n}\mathbb{E}\left[\mathbf{1}\{x_n \ge x/2\}\frac{\overline{\nu}(x_n)\tilde{K}(2x_n, T_0)}{x_n}\right] + \varepsilon_n(x).$$

Starting at this point, the treatment is very similar to that following (3.17). First, by

$$\begin{aligned}\frac{x}{A(x)} &\sum_{L(x)\le n\le A(\delta x)} \mathbb{P}(\Omega_{n,x}) \\ &\le \frac{C_\nu T_0}{A(x)^2}\sum_{n\le A(\delta x)} n + C_\nu \tilde{R}(x) + \frac{x}{A(x)}\sum_{L(x)\le n\le A(\delta x)} \varepsilon_n(x) \\ &= O(\delta^2) + C_\nu \tilde{R}(x) + o(1),\end{aligned}$$

where writing $\theta = 1/\alpha - 1$,

$$\tilde{R}(x) = \frac{x}{A(x)}\sum_{L(x)\le n\le A(\delta x)} \frac{n^{-\theta}}{\beta(n)}\mathbb{E}\left[\mathbf{1}\{x_n \ge x/2\}\frac{\overline{\nu}(x_n)\tilde{K}(2x_n, T_0)}{x_n}\right].$$

If $\alpha \in (0, 1/2)$, then by Assumption 1, (2.3) holds for $\tilde{K}(x, T_0)$, and hence

$$\begin{aligned}\tilde{R}(x) &= \frac{x}{A(x)} o\left(\sum_{L(x)\le n\le A(\delta x)} \frac{n^{-\theta}}{\beta(n)}\mathbb{E}\left[\mathbf{1}\{x_n \ge x/2\}\frac{\overline{\nu}(x_n)A(2x_n)^2}{x_n u_\theta(2x_n)}\right]\right) \\ &= \frac{x}{A(x)} o\left(\sum_{L(x)\le n\le A(\delta x)} \frac{n^{-\theta}}{\beta(n)}\frac{A(x)}{x u_\theta(x)}\right) = o(1), \qquad x \to \infty.\end{aligned}$$

If $\alpha = 1/2$, then by Assumption 1, (2.4) holds for $\tilde{K}(x, T_0)$. As a result, if $\tilde{u}(x)/\tilde{u}(L(x)) \to 1$, then

$$\tilde{R}(x) = O\biggl(\sum_{L(x)\le n\le A(\delta x)} \frac{n^{-\theta}}{\beta(n)} \mathbb{E}\biggl[\mathbf{1}\{x_n \ge x/2\} \frac{x_n}{A(x_n)} \cdot \frac{\overline{\nu}(x_n) A(2x_n)^2}{x_n \tilde{u}(2x_n)}\biggr]\biggr)$$

$$= O\biggl(\sum_{L(x)\le n\le A(\delta x)} \frac{n^{-\theta}}{\beta(n)} \frac{1}{\tilde{u}(x)}\biggr) = o(1), \qquad x \to \infty.$$

The case $\tilde{u}(x)/\tilde{u}(L(x)) \not\to 1$ can be shown likewise. This then completes the proof of Theorem 2.5.

PROOF OF LEMMA 3.5. By Proposition 2.3, it suffices to show that as $x \to \infty$, $\mathbb{P}\{X \in x + I\} = O(\overline{F}(x)/x^{1-\kappa})$. For $x > 0$,

$$\begin{aligned}
&\mathbb{P}\{X \in x + I\} \\
&\quad = \mathbb{P}\{\zeta_N + W \in x + I\} \\
&\quad \le \mathbb{P}\{\zeta_N + W \in x + I, N < \ln x, W < x/2\} + \mathbb{P}\{W \ge x/2\} \\
&\qquad + \mathbb{P}\{N \ge \ln x\} \\
&\quad \le \sum_{n<\ln x} \frac{e^{-\nu_0}\nu_0^n}{n!} \sup_{z>x/2} \mathbb{P}\{\zeta_n \in z + I\} + \mathbb{P}\{W \ge x/2\} + \mathbb{P}\{N \ge \ln x\}.
\end{aligned}$$

Given $\gamma \in (0, 1)$, by $x^\gamma \ln x = o(x)$, for $x \gg 1$, $n < \ln x$ and $z > x/2$, if $\zeta_n \in z + I$, then there is at least one $1 \le i \le n$ with $Y_i > z^\gamma$, and if there is exactly one such i, then $Y_i > z/2$. Thus

$$\begin{aligned}
&\mathbb{P}\{\zeta_n \in z + I\} \\
&\quad \le n^2 \mathbb{P}\bigl\{\zeta_n \in z + I, Y_{n-1} > z^\gamma, Y_n > z^\gamma\bigr\} + n\mathbb{P}\{\zeta_n \in z + I, Y_n > z/2\}.
\end{aligned}$$

First, following the argument to bound $E^{(3)}_{n,x}$ in the proof of Lemma 3.1,

$$\begin{aligned}
&\mathbb{P}\bigl\{\zeta_n \in z + I, Y_{n-1} > z^\gamma, Y_n > z^\gamma\bigr\} \\
&\quad = \sum_{k=0}^{\infty} \mathbb{P}\bigl\{\zeta_n \in z + I, Y_{n-1} > z^\gamma, Y_n \in z^\gamma + kh + I\bigr\} \\
&\quad \le \sup_{t>z^\gamma} \mathbb{P}\{Y \in t + I\} \sum_{k=0}^{\infty} \mathbb{P}\bigl\{\zeta_{n-1} \in z - z^\gamma - kh + J, Y_{n-1} > z^\gamma\bigr\} \\
&\quad \le 2\mathbb{P}\bigl\{Y > z^\gamma\bigr\} \sup_{t>z^\gamma} \mathbb{P}\{Y \in t + I\}.
\end{aligned}$$

By $\mathbb{P}\{Y > z^\gamma\} = \overline{\nu}(z^\gamma)/\nu_0$ and $\mathbb{P}\{Y \in t + I\} = \nu(t + I)/\nu_0 = O(\overline{\nu}(t)/t^{1-\kappa})$, the RHS of the display is $O(\overline{\nu}(z^\gamma)^2/z^{\gamma(1-\kappa)}) = O(\overline{\nu}(x^\gamma)^2/x^{\gamma(1-\kappa)})$. It follows that if

$\gamma > (\alpha + 1 - \kappa)/(2\alpha + 1 - \kappa)$, then the RHS is $o(\overline{F}(x)/x^{1-\kappa})$. With a similar argument, we also get

$$\mathbb{P}\{\zeta_n \in z + I, Y_n > z/2\} \le 2 \sup_{t>z/2} \mathbb{P}\{Y \in z/2 + I\} = O\big(\overline{F}(x)/x^{1-\kappa}\big).$$

As a result,

$$\sum_{n<\ln x} \frac{e^{-\nu_0}\nu_0^n}{n!} \sup_{z>x/2} \mathbb{P}\{\zeta_n \in z + I\} = O\big(\mathbb{E}N^2 \cdot \overline{F}(x)/x^{1-\kappa}\big)$$
$$= O\big(\overline{F}(x)/x^{1-\kappa}\big).$$

On the other hand, $\mathbb{P}\{W \ge x/2\} \le \mathbb{E}[e^{2(W-x/2)}] = O(e^{-x})$ and, for any $M > 0$, $\mathbb{P}\{N \ge \ln x\} \le \mathbb{E}[e^{M(N-\ln x)}] = O(x^{-M})$. By letting $M > \alpha + 1 - \kappa$, the above bounds together yield $\mathbb{P}\{X \in x + I\} = O(\overline{F}(x)/x^{1-\kappa})$, as desired. □

APPENDIX

In Section 2, we remarked that if the SRT holds, then (2.2) holds for any $L(x) = o(A(x))$. This follows from the following

PROPOSITION A. *For F satisfying both* (1.2) *and* (1.3),

$$\liminf_{x\to\infty} x\overline{F}(x)U(x + I) = h\Lambda_F, \tag{A.1}$$

where Λ_F is defined in (2.5), *and $h > 0$ is arbitrary if F is nonarithmetic and is the span of F otherwise.*

Indeed, if the SRT holds, then lim inf in (A.1) can be replaced with lim. On the other hand, by Lemma 3.4,

$$\lim_{\delta\to 0}\lim_{x\to\infty} x\overline{F}(x) \sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in x + I\} = h\Lambda_F, \tag{A.2}$$

where $J_\delta(x) = (A(\delta x), A(x/\delta))$. It then follows that

$$\lim_{\delta\to 0}\lim_{x\to\infty} x\overline{F}(x) \sum_{n\le A(\delta x)} \mathbb{P}\{S_n \in x + I\} = 0$$

and hence (2.2) holds for any $L(x) = o(A(x))$.

PROOF OF PROPOSITION A. It is well known that if F is nonlattice with support in $[0, \infty)$ and infinite mean, then Proposition A holds ([1], Theorem 8.6.6). For the general case, we follow the proof in [1]. Denoting $V(x) = U((0, x^+])$, the starting point is the identity

$$\lim_{x\to\infty} \overline{F}(x)V(x) = \Lambda_F/\alpha. \tag{A.3}$$

This is established on page 361 in [1]. However, the proof there relies on the Laplace transforms of F and U, so it cannot apply to the general case as the transforms may be ∞. Instead, we shall prove (A.3) using a more probabilistic argument, which is basically a coarse version of the one for the SRT. For now assume (A.3) to be true. Then (A.2) implies

$$\liminf_{x\to\infty} x\overline{F}(x)U(x+I) \geq h\Lambda_F.$$

Assume that strict inequality holds. Then by (A.3), there is $h' > h$, such that for all $x \gg 1$, $U(x+I) \geq h'\alpha V(x)/x$. Also $V \in \mathcal{R}_\alpha$. Then as $t \to \infty$,

$$\int_0^t U(x+I)\,\mathrm{d}x \geq \big(1+o(1)\big)h'\alpha \int_0^t V(x)x^{-1}\,\mathrm{d}x \sim h'V(t).$$

However, since $U(x+I) = V(x+h) - V(x)$ for $x \geq 0$, LHS $= \int_t^{t+h} V - \int_0^h V \sim hV(t)$, which contradicts the above display. Thus (A.1) follows.

It remains to show (A.3). Given $\delta \in (0,1)$,

$$\sum_{n\leq A(\delta x)} \mathbb{P}\{S_n \in (0,x]\} \leq A(\delta x) \sim \delta^\alpha A(x), \qquad x \to \infty. \tag{A.4}$$

On the other hand, by the LLTs and the boundedness of g, there is $C > 0$, such that for all $x \gg 1$, $n \geq A(x/\delta)$, and $t \in \mathbb{R}$, $\mathbb{P}\{S_n \in t+I\} \leq C/a_n$. Then, by dividing $(0,x]$ into $\lceil x/h \rceil$ intervals of equal length, it is seen that $\mathbb{P}\{S_n \in (0,x]\} \leq Cx/a_n$. Consequently,

$$\begin{aligned}\sum_{n\geq A(x/\delta)} \mathbb{P}\{S_n \in (0,x]\} &\leq Cx \sum_{n\geq A(x/\delta)} \frac{1}{a_n} \\ &\sim \frac{C'xA(x/\delta)}{x/\delta} \sim C''\delta^{1-\alpha}A(x).\end{aligned} \tag{A.5}$$

By the central limit theorem, as $n \to \infty$, $G_n(s) := \mathbb{P}\{0 < S_n/a_n \leq s\} \to G(s) := \int_0^{s^+} g$ for each s. Since G_n and G are nondecreasing functions with range contained in $[0,1]$, and G is continuous, the pointwise convergence gives $\sup|G_n - G| \to 0$. Then by $\mathbb{P}\{S_n \in (0,x]\} = G_n(x/a_n)$, as $x \to \infty$,

$$\begin{aligned}&\sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in (0,x]\} \\ &\quad= \sum_{n\in J_\delta(x)} G(x/a_n) + \big[A(x/\delta) - A(\delta x)\big]o(1) \\ &\quad= \int_{A(\delta x)}^{A(x/\delta)} G\big(x/A^{-1}(t)\big)\,\mathrm{d}t + \big(\delta^{-\alpha} - \delta^\alpha\big)o(1)A(x).\end{aligned}$$

By change of variable $u = x/A^{-1}(t)$ and $A'(u) \sim \alpha A(u)/u$ as $u \to \infty$,

$$\int_{A(\delta x)}^{A(x/\delta)} G\big(x/A^{-1}(t)\big)\,\mathrm{d}t = \int_\delta^{1/\delta} G(u)\frac{x}{u^2}A'(x/u)\,\mathrm{d}u$$
$$\sim \int_\delta^{1/\delta} G(u)\frac{\alpha A(x/u)}{u}\,\mathrm{d}u$$
$$\sim A(x)\int_\delta^{1/\delta} G(u)\alpha u^{-1-\alpha}\,\mathrm{d}u, \qquad x \to \infty.$$

As a result,

$$\lim_{\delta\to 0}\lim_{x\to\infty}\frac{1}{A(x)}\sum_{n\in J_\delta(x)} \mathbb{P}\{S_n \in (0,x]\} = \int_0^\infty G(u)\alpha u^{-1-\alpha}\,\mathrm{d}u. \tag{A.6}$$

The RHS is Λ_F/α. Combining (A.4)–(A.6), then (A.3) follows. □

Acknowledgments. The author would like to thank the referees for careful reviews and useful suggestions.

REFERENCES

[1] BINGHAM, N. H., GOLDIE, C. M. and TEUGELS, J. L. (1989). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press, Cambridge. MR1015093

[2] BREIMAN, L. (1992). *Probability*. *Classics in Applied Mathematics* **7**. SIAM, Philadelphia, PA. MR1163370

[3] DENISOV, D., DIEKER, A. B. and SHNEER, V. (2008). Large deviations for random walks under subexponentiality: The big-jump domain. *Ann. Probab.* **36** 1946–1991. MR2440928

[4] DONEY, R. A. (1997). One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107** 451–465. MR1440141

[5] DONEY, R. A. (2012). Local behaviour of first passage probabilities. *Probab. Theory Related Fields* **152** 559–588. MR2892956

[6] DONEY, R. A. and GREENWOOD, P. E. (1993). On the joint distribution of ladder variables of random walk. *Probab. Theory Related Fields* **94** 457–472. MR1201554

[7] ERICKSON, K. B. (1970). Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** 263–291. MR0268976

[8] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*. *Vol. II*, 2nd ed. Wiley, New York. MR0270403

[9] GARSIA, A. and LAMPERTI, J. (1962/1963). A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37** 221–234. MR0148121

[10] GOUËZEL, S. (2011). Correlation asymptotics from large deviations in dynamical systems with infinite measure. *Colloq. Math.* **125** 193–212. MR2871313

[11] HULT, H., LINDSKOG, F., MIKOSCH, T. and SAMORODNITSKY, G. (2005). Functional large deviations for multivariate regularly varying random walks. *Ann. Appl. Probab.* **15** 2651–2680. MR2187307

[12] KOREVAAR, J. (2004). *Tauberian Theory*: *A Century of Developments*. *Grundlehren der Mathematischen Wissenschaften* **329**. Springer, Berlin. MR2073637

[13] PETROV, V. V. (1995). *Limit Theorems of Probability Theory*: *Sequences of Independent Random Variables*. *Oxford Studies in Probability* **4**. Oxford Univ. Press, New York. MR1353441

[14] ROGOZIN, B. A. (1971). Distribution of the first laddar moment and height, and fluctuations of a random walk. *Theory Probab*. *Appl*. **16** 539–613.

[15] SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR1739520

[16] TANG, Q. (2006). Insensitivity to negative dependence of the asymptotic behavior of precise large deviations. *Electron*. *J*. *Probab*. **11** 107–120 (electronic). MR2217811

[17] TAO, T. and VU, V. (2006). *Additive Combinatorics*. *Cambridge Studies in Advanced Mathematics* **105**. Cambridge Univ. Press, Cambridge. MR2289012

[18] VATUTIN, V. A. and TOPCHII, V. A. (2013). A key renewal theorem for heavy tail distributions with $\beta \in (0, 0.5]$. *Theory Probab*. *Appl*. **58** 387–396.

[19] WALTERS, P. (1982). *An Introduction to Ergodic Theory*. *Graduate Texts in Mathematics* **79**. Springer, New York. MR0648108

[20] WILLIAMSON, J. A. (1968). Random walks and Riesz kernels. *Pacific J*. *Math*. **25** 393–415. MR0226741

DEPARTMENT OF STATISTICS
UNIVERSITY OF CONNECTICUT
215 GLENBROOK ROAD, U-4120
STORRS, CONNECTICUT 06269
USA
E-MAIL: zhiyi.chi@uconn.edu