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Construction of stationary self-similar generalized fields by random wavelet expansion

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Abstract. Random wavelet expansion is introduced in the study of stationary self-similar generalized random fields. It is motivated by a model of natural images, in which 2D views of objects are randomly scaled and translated because the objects are randomly distributed in the 3D space. It is demonstrated that any stationary self-similar random field defined on the dual space of a Schwartz space of smooth rapidly decreasing functions has a random wavelet expansion representation. To explicitly construct stationary self-similar random fields, random wavelet expansion representations incorporating random functionals of the following three types are considered: (1) a multiple stochastic integral over the product domain of scale and translate, (2) an iterated one, first integrating over the scale domain, and (3) an iterated one, first integrating over the translate domain. We show that random wavelet expansion gives rise to a variety of stationary self-similar random fields, including such well-known processes as the linear fractional stable motions.

1. Introduction

Self-similar distributions (**ssd's** henceforth) form an intensively studied area in probability theory [21–23, 4, 5, 20, 1, 19, 18, 2]. Interest in such distributions originated in physics, especially renormalization group theory and critical phenomena. Recently ssd's have also gained interest in human vision [16, 17, 11, 6] and image analysis [24, 3, 8, 14].

This article is mainly concerned with stationary self-similar generalized random fields. Dobrushin [4] gave a complete description of Gaussian ssd's and also established a representation of ssd's subordinated to Gaussian ones, all of which have finite variances. Taqqu and Wolpert [23] and Maejima [12] investigated the construction of infinite variance self-similar processes with stationary increments subordinated to Poisson measures. These processes, when differentiated, yield ssd's.

Whereas the above methods are established for specific types of ssd's, the approach presented here is a general one. Termed "random wavelet expansion", it

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was motivated by image analysis. It has been noted that natural images exhibit strong scaling property in addition to stationarity. Therefore it is sensible to build stationary self-similar image models. However, regular functions sampled from ssd's are either constant or fractal, which is in contrast with the rich structure natural images have. For this reason, Mumford [13] proposed that images are better modeled as generalized functions. As far as the author knows, Mumford and Gidas [14] obtained the first random wavelet expansion representation as follows,

$$I(x, y) = \sum_i J_i(\lambda_i x + a_i, \lambda_i y + b_i), \tag{1.1}$$

where $\{(\lambda_i, a_i, b_i)\}$ is a random sample from a Poisson point process on $\mathbb{R} \times \mathbb{R}^2$ with intensity measure $\lambda^{-1}\, d\lambda\, da\, db$ and J_i are independent random generalized functions from some distribution ν_0. This representation was derived from a statistical model on the composition of the object surfaces in natural images. Around the same time, another image model was proposed in [3] from the perspective that natural images are generated by projecting the 3D space onto an image film. Since objects are randomly located in the 3D space, their 2D views are randomly scaled and translated, leading to (1.1). Section 2 gives more detail on the model and its connection with (1.1).

The representation (1.1) will be generalized, while still termed "random wavelet expansion". A random wavelet expansion representation has the form $\langle I, \phi\rangle = \langle W, \Psi_{g,H}\phi\rangle$, where ϕ is a test function on $\mathbb{R}^n$, W a linear random functional, and $\Psi_{g,H}$ a (deterministic) continuous wavelet expansion operator with mother wavelet g and index H. Then $(\Psi_{g,H}\phi)(u, v)$ is a function on $\mathbb{R} \times \mathbb{R}^n$, with $u \in \mathbb{R}$ corresponding to the scale, and $v \in \mathbb{R}^n$ the translate. The physical meaning of the above formula can be seen more clearly in light of the model in section 2.

Random wavelet expansion is general in the following sense: any ssd defined on the dual space of a Schwartz space of smooth rapidly decreasing functions has a random wavelet expansion representation $I = \Psi^*_{g,H} W$, with g an infinitely differentiable function with compact support (Theorem 2). Nevertheless, the result gives little information on how to characterize a ssd by random wavelet expansion. To get such information, it is necessary to have ssd's explicitly constructed.

For the random functional W in the representation $I = \Psi^*_{g,H} W$, the most natural choice is stochastic integrals. As $\Psi_{g,H}\phi$ is a function of scale and translate, W can be a multiple integral over the product domain of the two (scale-translate domain); or an iterated integral that first integrates over the translate domain, then over the scale domain (Theorem 3); or an iterated integral that integrates in the other order (Theorem 7). These possibilities are investigated in detail in sections 4–6.

In section 4, after setting up the framework of random wavelet expansion, we will explore the idea of using stochastic integrals that first integrate over the translate domain, to yield a quite general construction scheme. Intuitively, the scheme is to "stack" along the scale domain different types of stationary random fields, each defined on the translate domain. If the locations and the types of the stacked random fields form a stationary point process, then under certain independence condition for these random fields, the superimposition of their "projections" along the scale domain is a ssd. The idea of the scheme also enables us to better understand the spatial structure underlying self-similarity. In section 4.3 we consider the relationship between symmetry and self-similarity. We will show that, under certain minor conditions, a stationary random field is self-similar if it is self-similar with respect to both symmetric test functions and anti-symmetric ones (Theorem 4). On the other hand, by employing a stochastic integral that is *not* stationary along the scale domain, we obtain stationary random fields that are self-similar with respect to symmetric test functions while not to anti-symmetric ones, or vice versa. The results demonstrate that the stationarity of a random integral along the scale domain plays a non-trivial role in ensuring self-similarity.

Section 5 studies in detail random wavelet expansion representations that incorporate multiple stochastic integrals. All the ssd's in this section are constructed based on a simple fact. That is, random wavelet expansion converts scalings and translations into diffeomorphisms of the scale-translate domain that have Jacobian 1. As will be seen, as Poisson random measures, stable symmetric random measures, and many other random measures that underlie infinitely divisible random fields are invariant under such diffeomorphisms, they can be combined with random wavelet expansion to get ssd's. A rich class of stable symmetric ssd's can be constructed in this way. Briefly, if M is an α-stable symmetric measure, $\alpha \in (0, 2)$, then for representations $\langle I_g,\ \phi\rangle = \int(\Psi_{g,H}\phi)\, dM$ and $\langle I_h,\ \phi\rangle = \int(\Psi_{h,H}\phi)\, dM$, the necessary and sufficient condition for $I_g \overset{\mathcal{D}}{=} I_h$ is that g and h be scaled and translated versions of each other, up to a sign (Theorem 5).

In section 6, we consider the case where W is an iterated stochastic integral that first integrates over the scale domain. The integral will be combined with a "dual" operator $L_{g,H}$ of $\Psi_{g,H}$ (see Eq. (3.3)). Random wavelet expansion in this section gives rise to the familiar linear fractional or log-linear stable motions on $\mathbb{R}$ [18], giving these random processes an underlying spatial structure. First, it will be demonstrated that letting $\phi = \mathbf{1}_{[0,t]}$, $t \in \mathbb{R}$, the process $X_t = \langle W,\ L_{g,H}\mathbf{1}_{[0,t]}\rangle$ can be a linear fractional stable motion, a log-linear fractional stable motion, or simply a Lévy motion (Proposition 8). This formally argues that these motions arise from the random wavelet expansion representation $I = L^*_{g,H}W$. On the other hand, as I can not be directly imposed on an indicator function, to make the argument precise, we will show that X_t is the limit in distribution of $\langle I,\ \phi_n\rangle$ for a sequence of smooth rapidly decreasing functions ϕ_n (Theorem 8). Such limiting procedure is called "discretization" in some literature [4].

The other parts of the article are organized as the following. Section 3 fixes notation, defines the expansion operator $\Psi_{g,H}$ and collects some of its important functional properties. In particular, the dual operator of $\Psi_{g,H}$ will be quite useful (Lemma 1). Section 7 gives proofs of the technical results on functional properties of $\Psi_{g,H}$. Finally, section 8 discusses possible generalization of random wavelet expansion to random fields invariant under transformations that consist a finitely generated commutative Lie group.

2. An image analysis background

In this section we present an image analysis background for random wavelet expansion, to illustrate how the latter naturally arises from statistical modeling of natural images.

One remarkable empirical property of natural images is their apparent statistical scale invariance. That is, the marginal distributions of many statistics of natural images are invariant when the images are scaled. This *natural* phenomenon is of interest in biological vision as well as in computer vision.

The statistical model devised in [3] to explain the origin of scaling of natural images regards such images as 2D views of the 3D world through the lens of a camera. The model is based on several assumptions:

(1) The objects are simplified as planar templates parallel to the camera image plane. Each object has a fixed reference point, so that its spatial location refers to the location of its reference point;
(2) Assuming sparseness of the objects in the 3D world, a natural image is approximately the arithmetic sum of the 2D views of the objects;
(3) The locations of the objects form a point process in the 3D space;
(4) The shape of an object is independent of its spatial location.

Set up appropriate coordinate systems in the 3D space and on the camera image plane so that a point with coordinates (λ, y, z) in space is at distance λ from the camera lens and its projection on the image plane is at $(-y/\lambda, -z/\lambda)$.

Given an object, let $f(\xi, \eta)$ be its "standard view", which is a function describing the light or color intensity inside its 2D view, when it is located at $(1, 0, 0)$. Then by projective geometry, if the object is moved to (λ, y, z), $\lambda > 0$, the intensity inside its 2D view is changed to $f(\lambda\xi + y, \lambda\eta + z)$. Letting (λ_i, y_i, z_i) be the locations of the objects, and f_i the standard views of the objects, the above assumptions of the model imply that an image I can be written as $I(\xi, \eta) = \sum_i f_i(\lambda_i\xi + y_i, \lambda_i\eta + z_i)$, with $\{f_i\}$ independent of $\{(\lambda_i, y_i, z_i)\}$, which is exactly (1.1). Due to the obvious random scaling and translation of the template functions f_i in its summation, (1.1) is called "random wavelet expansion" in [14]. It is a simple matter to verify that if $\{(\lambda_i, y_i, z_i)\}$ is a random sample from a Poisson point process with intensity measure $\lambda^{-1}\, d\lambda\, dy\, dz$, then, formally, I is stationary as well as scale invariant.

In accord with the view of [13], the image I is considered a generalized function. As a mathematical simplification, assume the templates are identical and let g be the standard view of the templates. Then for any test function ϕ,

$$\langle I, \phi \rangle = \sum_i \int g(\lambda_i\xi - y_i, \lambda_i\eta - z_i)\, \phi(\xi, \eta)\, d\xi\, d\eta. \tag{2.1}$$

Let $\lambda_i = e^{u_i}$, $v_i = -(y_i, z_i)$, $x = (\xi, \eta)$. Define function $\Psi_g\phi : (u, v) \to \int g(e^u x + v)\phi(x)\, dx$, then $\langle I, \phi\rangle = \langle W, \Psi_g\phi\rangle$, with $W = \sum_i \delta(u - u_i)\delta(v - v_i)$. Then I has the (formal) representation

$$I = \Psi_g^* W. \tag{2.2}$$

For the representation (2.1), I has to be composed of countably many randomly scaled and translated copies of g. The representation (2.2) on the other hand dispenses with this requirement, hence allowing generalization. Since the essence of (2.2) is the combination of a random functional (W) and a (continuous) deterministic expansion (Ψ_g, with the index $H = 1$ omitted), we will still call it "random wavelet expansion".

3. Preliminaries

3.1. Notation

A multiple index is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in \mathbb{N} \cup \{0\}$. Denote $|\alpha| = \sum \alpha_i$. Given $x \in \mathbb{R}^n$ and $\phi \in C^\infty(\mathbb{R}^n)$, denote

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial_\alpha \phi(x) = \frac{\partial^{|\alpha|}\phi(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of real-valued smooth rapidly decreasing functions on $\mathbb{R}^n$ ([7], pp 20–23). Denote

$$C_0^\infty(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \phi \text{ has compact support}\}$$
$$\mathcal{S}_k(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \int x^\alpha \phi(x)\, dx = 0,\ |\alpha| < k\}, \quad k \geq 1.$$

The topology of $\mathcal{S}_k(\mathbb{R}^n)$ is inherited from $\mathcal{S}(\mathbb{R}^n)$. For convenience, we will occasionally use $\mathcal{S}_0(\mathbb{R}^n)$ for $\mathcal{S}(\mathbb{R}^n)$. All $S_k(\mathbb{R}^n)$ are nuclear linear topological spaces ([7], pp 73 & 86). By convention, $\mathcal{S}'_k(\mathbb{R}^n)$ stands for the space of continuous linear functionals on $\mathcal{S}_k(\mathbb{R}^n)$, equipped with the weak topology. The σ-algebra of Borel subsets of $\mathcal{S}'_k(\mathbb{R}^n)$ with respect to the weak topology is denoted by $\mathcal{B}(\mathcal{S}'_k(\mathbb{R}^n))$. For $F \in \mathcal{S}'_k(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ we often write $\langle F, \phi\rangle$ for $F(\phi)$.

We will refer to linear random functional on topological space as random functional. Given $\mathbb{R}^n$, following [4], a generalized random process ($n = 1$) or field ($n > 1$) on $\mathbb{R}^n$ is a probability measure μ on $(\mathcal{S}'_k(\mathbb{R}^n), \mathcal{B})$, for some $k \geq 0$, with $\mathcal{B}$ the weak topology generated by cylinder sets. A random functional F on $\mathcal{S}_k(\mathbb{R}^n)$ is called a representation of μ, if the c.f. of F is the same as that of μ, i.e., $E[e^{i\langle F, \phi\rangle}] = \int e^{i\langle \omega, \phi\rangle} \mu(d\omega)$, $\phi \in \mathcal{S}_k(\mathbb{R}^n)$.

Given $t > 0$ and $v \in \mathbb{R}^n$, define scaling operator S_t and translation operator T_v, such that for any regular function ϕ on $\mathbb{R}^n$,

$$(S_t\phi)(x) = t^{-n}\phi(t^{-1}x), \quad (T_v\phi)(x) = \phi(x - v).$$

Denote by S_t^* and T_v^* the adjoint operators of S_t and T_v on $\mathcal{S}'_k(\mathbb{R}^n)$. Notice that if $F \in \mathcal{S}'_k(\mathbb{R}^n)$ happens to be a regular function, then $S_t^* F$ is a regular function with $(S_t^* F)(x) = F(tx)$ and $T_v^* F$ is another regular function with $(T_v^* F)(x) = F(x + v)$.

Definition 1. *Fix $H \in \mathbb{R}$. A probability measure μ on $\mathcal{S}'_k(\mathbb{R}^n)$ is called stationary and self-similar with index H (H-ss) if*

$$\mu(A) = \mu((t^H S_t^*)^{-1} A), \quad \mu(A) = \mu((T_v^*)^{-1} A), \quad t > 0, \quad v \in \mathbb{R}^n, \\ A \in \mathcal{B}(\mathcal{S}'_k(\mathbb{R}^n)). \tag{3.1}$$

If μ satisfies (3.1), then we call it an H-ssd.

By Bochner's theorem for nuclear spaces [7], if F is a random functional representation of μ, then μ is H-ss iff $\phi \in \mathcal{S}_k$, $\langle F, \phi\rangle \overset{\mathcal{D}}{=} \langle F, t^H S_t T_v \phi\rangle$. Formally, this can be regarded as $F(x) \overset{\mathcal{D}}{=} t^H F(tx + v)$.

3.2. Definition and functional properties of $\Psi_{g,H}$

Definition 2. *Given a function g and $H \in \mathbb{R}$, wavelet expansion operator $\Psi_{g,H}$ on $\mathcal{S}_k(\mathbb{R}^n)$, with "mother wavelet" g and index H, is defined by*

$$(\Psi_{g,H}\phi)(u, v) = \int_{\mathbb{R}^n} e^{Hu} g(e^u x + v)\phi(x)\, dx, \quad \phi \in \mathcal{S}_k(\mathbb{R}^n). \tag{3.2}$$

Comparing with (2.1), it is seen that (e^u, v) can be thought of as the spatial location of an object, with e^u the distance of the object from the camera lens, and v the shift of the object in the 3D space, parallel to the image plane. Furthermore, $e^{-u}v$ is the translate of the 2D view of the object on the image plane.

The following obvious duality on $\Psi_{g,H}$ is useful in proving several properties of $\Psi_{g,H}$, and also useful in the construction of α-stable fractional fields in section 6.

Lemma 1. *Suppose $g, \phi \in \mathcal{S}(\mathbb{R}^n)$. Then for any $H \in \mathbb{R}$,*

$$(\Psi_{g,H}\phi)(u, v) = (\Psi_{\phi,n-H}\, g)(-u, -e^{-u}v). \tag{3.3}$$

Lemma 2. *Fix $k, l \in \{0\} \cup \mathbb{N}$, $H \in \mathbb{R}$, and $p \in (0, \infty]$, such that $p(n+k-H) > n$ and $l > -H$. Then*

(1) Given $g \in \mathcal{S}_k(\mathbb{R}^n)$, $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_l(\mathbb{R}^n)$ into $L^p(\mathbb{R} \times \mathbb{R}^n)$;
(2) On the other hand, given $\phi \in \mathcal{S}_l(\mathbb{R}^n)$, the map $g \to \Psi_{g,H}\phi$ is a continuous operator from $\mathcal{S}_k(\mathbb{R}^n)$ into $L^p(\mathbb{R} \times \mathbb{R}^n)$;
(3) Given any integer $m \geq \max\{k, l\}$, define

$$h(u, v) = \frac{e^{(H-n-k)u}}{(1 + |e^{-u}v|)^m} \mathbf{1}_{[0,\infty)}(u) + \frac{e^{(H+l)u}}{(1 + |v|)^m} \mathbf{1}_{(-\infty,0)}(u). \tag{3.4}$$

There are functions $R \in C(\mathcal{S}_k(\mathbb{R}^n))$ and $Q \in C(\mathcal{S}_l(\mathbb{R}^n))$ with $R(0) = Q(0) = 0$, such that

$$|(\Psi_{g,H}\phi)(u, v)| \leq R(g)Q(\phi)h(u, v), \tag{3.5}$$

(4) *If* $m \geq \max\{k, l, n+k-H\}$ *(by assumption, this implies* $m > n/p$*), then for* h *defined by (3.4),* $h \in L^p(\mathbb{R} \times \mathbb{R}^n)$*, and* $F(v) = \int_{-\infty}^{\infty} h(u, v)\, du \in C(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Lemma 3. *Given any* $0 \neq g \in C_0^{\infty}(\mathbb{R}^n)$ *and* $H \in \mathbb{R}$, $\Psi_{g,H}$ *is a 1-1 map from* $\mathcal{S}(\mathbb{R}^n)$ *into* $C^{\infty}(\mathbb{R} \times \mathbb{R}^n)$.

Proofs of Lemma 2 and Lemma 3 are put off until section 7.

4. Random Wavelet Expansion

4.1. Definition and existence of random wavelet expansion representation for ssd

With Ψ_g replaced by $\Psi_{g,H}$, where $g \in \mathcal{S}_k(\mathbb{R}^n)$ for certain $k \in \{0\} \cup \mathbb{N}$ and $H \in \mathbb{R}$, representation (2.2) is now modified to

$$I = \Psi_{g,H}^* W. \tag{4.1}$$

We term (4.1) random wavelet expansion. The operator $\Psi_{g,H}$ is related to scaling and translation by the next lemma, which can be easily proved using

$$(T_{v_0} S_{e^{u_0}} \phi)(x) = (S_{e^{u_0}} \phi)(x - v_0) = e^{-nu_0} \phi(e^{-u_0}(x - v_0))$$

as well as change of variable in integration.

Lemma 4. *For any* $u_0 \in \mathbb{R}$ *and* $v_0 \in \mathbb{R}^n$,

$$(\Psi_{g,H} T_{v_0} S_{e^{u_0}} \phi)(u, v) = e^{-Hu_0} (U_{u_0,v_0} \Psi_{g,H} \phi)(u, v). \tag{4.2}$$

where the operator U_{u_0,v_0} *is defined by*

$$(U_{u_0,v_0} f)(u, v) = f(u + u_0, v + e^u v_0). \tag{4.3}$$

for any regular function f *on* $\mathbb{R} \times \mathbb{R}^n$.

We will refer to any $U_{u,v}$ or $U_{u,v}^*$, if well-defined, as U-transform. Based on Lemma 4, we have a general result on (4.1), which is the principle idea of random wavelet expansion.

Theorem 1. *Suppose that*

(1) $\mathcal{F}$ *is a linear topological space of functions on* $\mathbb{R} \times \mathbb{R}^n$*;*
(2) W *is a random functional on* $\mathcal{F}$ *such that it is invariant under* $U_{u,v}^*$ *for any* (u, v) *and such that its c.f.* C *is continuous on* $\mathcal{F}$*;*
(3) $\Psi_{g,H}$ *is a continuous operator from* $\mathcal{S}_k(\mathbb{R}^n)$ *to* $\mathcal{F}$.

Then $C \circ \Psi_{g,H}$ *is the c.f. of an* H*-ssd on* $\mathcal{S}_k'(\mathbb{R}^n)$.

Proof. Let $L = C \circ \Psi_{g,H}$. Then clearly $L(0) = C(0) = 1$. Because $\Psi_{g,H}$ is linear and C is positive definite, L is also positive definite. Condition (3) as well as the continuity of C implies the continuity of L. Since $\mathcal{S}(\mathbb{R}^n)$ is nuclear, by Theorem 2 on p.350 of [7], L is the c.f. of a probability distribution on $\mathcal{S}_k'(\mathbb{R}^n)$.

Since C is invariant under $U_{u,v}^*$ for any $(u, v) \in \mathbb{R} \times \mathbb{R}^n$, then

$$L(e^{Hu} T_v S_{e^u} \phi) = C(U_{u,v} \Psi_{g,H} \phi) = C(\Psi_{g,H} \phi) = L(\phi), \quad \phi \in \mathcal{S}_k'(\mathbb{R}^n).$$

Therefore, the probability distribution determined by L is H-ss. □

Theorem 1 gives an effective way to construct ssd's. On the other hand, it raises the question whether such $\mathcal{F}$ and W exist. The following result, which in essence is a tautology, shows that for any ssd on $\mathcal{S}_k'(\mathbb{R}^n)$, we can find a random wavelet expansion which gives rise to the random field.

Theorem 2. *Given $l \in \{0\} \cup \mathbb{N}$, suppose μ is an H-ssd on $\mathcal{S}_l'(\mathbb{R}^n)$. Then for any $g \in C_0^\infty(\mathbb{R}^n)$, there is a linear topological space $\mathcal{F}$ of functions on $\mathbb{R} \times \mathbb{R}^n$ as well as a random functional W on $\mathcal{F}$, such that μ admits the random wavelet expansion representation (4.1).*

Proof. Write $\mathcal{E} = \mathcal{S}_l(\mathbb{R}^n)$. By Lemma 3, $\Psi_{g,H}$ is a 1-1 linear map from $\mathcal{E}$ into $C^\infty(\mathbb{R} \times \mathbb{R}^n)$. The range of $\Psi_{g,H}$, i.e., $\Omega = \Psi_{g,H}(\mathcal{E})$, is a linear space. Introduce a topology $\mathcal{T}$ on Ω such that $A \in \mathcal{T}$ if and only if $\Psi_{g,H}^{-1}(A)$ is an open set in $\mathcal{E}$. Then $\mathcal{F} = (\Omega, \mathcal{T})$ is a linear topological space isomorphic to $\mathcal{E}$, with isomorphism $\Psi_{g,H}$. Consequently, $(\mathcal{F}', \mathcal{B}(\mathcal{F}'))$ and $(\mathcal{E}', \mathcal{B}(\mathcal{E}'))$ are isomorphic as measurable spaces with isomorphism $\Psi_{g,H}^*$, with $\mathcal{B}(\mathcal{F}')$ and $\mathcal{B}(\mathcal{E}')$ the σ-algebras with respect to weak topologies of $\mathcal{F}'$ and $\mathcal{E}'$, respectively. Define ν on $\mathcal{F}'$ by $\nu(A) = \mu((\Psi_{g,H}^*)^{-1}(A))$. Let $W = \{\langle W,\ f\rangle : f \in \Omega\}$ be such that $\langle W,\ f\rangle(\omega) = \omega(f)$, $\omega \in \mathcal{F}'$. Then $I = \Psi_{g,H}^* W$ is a representation of μ. □

4.2. *"Stack-and-project" scheme for random wavelet expansion*

As discussed in section 1, the random functional W in random wavelet expansion (4.1) can be three types of stochastic integrals. In this subsection we explore those which first integrate over the translate domain. The consideration leads us to a general scheme as the following.

Suppose $\mathcal{E}$ is a linear topological space of functions on $\mathbb{R}^n$ and is closed under translation. Given set Λ, let $\mathcal{X} = \{X_\lambda, \lambda \in \Lambda\}$ be a family of real-valued random functionals on $\mathcal{E}$, with each X_λ being stationary, i.e., for $v \in \mathbb{R}^n$, $X_\lambda \overset{\mathcal{D}}{=} T_v^* X_\lambda$. Assume $\mathbb{Y}$ is a marked stationary point process on $\mathbb{R}$, with each mark in Λ. Denote by $\{(u_i, \lambda_i), i \in \mathbb{Z}\}$ a random sample from $\mathbf{Y}$, with $u_i \in \mathbb{R}$ the coordinate and $\lambda_i \in \Lambda$ the mark of the i-th point, respectively. Let $\{Z_i, i \in \mathbb{Z}\}$ be a random process, such that $Z_i \in \mathcal{X}$ and

$$\mathcal{L}(\{Z_i\}|\{u_i, \lambda_i\}) = \mathcal{L}(\{Z_i\}|\{\lambda_i\}) = \mathcal{L}(\{\tilde{X}_i\}), \tag{4.4}$$

with $\tilde{X}_i \overset{\mathcal{D}}{=} X_{\lambda_i}$ and $\tilde{X}_i$ independent of each other. Given function $f(u, v)$, $u \in \mathbb{R}$, $v \in \mathbb{R}^n$, let $f(u, \cdot)$ be the map $v \to f(u, v)$.

Theorem 3. *Suppose $\mathcal{F}$ is a linear topological space of functions on $\mathbb{R} \times \mathbb{R}^n$, such that $f(u, \cdot) \in \mathcal{E}$ for all $f \in \mathcal{F}$, $u \in \mathbb{R}$. Given $\mathbb{Y}$ and $\{Z_i\}$ as above, suppose for each $f \in \mathcal{F}$, with probability 1,*

$$\langle W,\ f\rangle = \sum_i \langle Z_i,\ f(u_i, \cdot)\rangle \tag{4.5}$$

is well-defined, and $E[e^{i\langle W,\ f\rangle}]$ is continuous on $\mathcal{F}$. If $\Psi_{g,H}$ is a continuous map from $\mathcal{S}_k(\mathbb{R}^n)$ to $\mathcal{F}$, then $I = \Psi_{g,H}^* W$ is a representation of an H-ssd on $\mathcal{S}_k'(\mathbb{R}^n)$.

Remark. From (4.5) it is seen that $\langle W,\ f\rangle$ is obtained by first getting $\langle Z_i,\ f(u_i,\cdot)\rangle$, which can be regarded as integration over the translate domain with u_i fixed, then taking the sum over all u_i, which corresponds to integration over the scale domain.

Proof. By Theorem 1, we only need to check that the distribution of W is invariant under $U^*_{s,0}$ and $U^*_{0,t}$. For the first one, since $(U_{s,0}f)(u,v) = f(u+s,v)$,

$$\langle U^*_{s,0}W,\ f\rangle = \langle W,\ U_{s,0}f\rangle = \sum_i \langle Z_i,\ f(u_i+s,\cdot)\rangle,$$

implying $U^*_{s,0}W$ is determined by the sequence $\{(u_i+s,\lambda_i), k\in\mathbb{Z}\}$ via (4.5). Since $\mathbb{Y}$ is stationary, $\{(u_i+s,\lambda_i)\} \stackrel{\mathcal{D}}{=} \{(u_i,\lambda_i), n\in\mathbb{Z}\}$, and thus $U^*_{s,0}W \stackrel{\mathcal{D}}{=} W$.

For invariance under $U^*_{0,t}$, since $(U_{0,t}f)(u,v) = f(u, v+e^u t)$, letting $t_i = e^{u_i}t$,

$$\langle U^*_{0,t}W,\ f\rangle = \langle W,\ U_{0,t}f\rangle = \sum_i \langle Z_i,\ (T_{t_i}f)(u_i,\cdot)\rangle = \sum_i \langle T^*_{t_i}Z_i,\ f(u_i,\cdot)\rangle.$$

Because by (4.4), conditional on $\{u_i,\lambda_i\}$, $\{Z_i\}$ is independent of $\{u_i\}$, therefore,

$$\begin{aligned}\sum_i \langle T^*_{t_i}Z_i,\ f(u_i,\cdot)\rangle &\underset{(a)}{\stackrel{\mathcal{D}}{=}} \sum_i \langle T^*_{t_i}\tilde{X}_i,\ f(u_i,\cdot)\rangle \underset{(b)}{\stackrel{\mathcal{D}}{=}} \sum_i \langle \tilde{X}_i,\ f(u_i,\cdot)\rangle \\ &\underset{(c)}{\stackrel{\mathcal{D}}{=}} \sum_i \langle Z_i,\ f(u_i,\cdot)\rangle,\end{aligned}$$

where (a) and (c) are due to (4.4), and (b) to the stationarity of all X_λ. This proves $U^*_{0,t}W \stackrel{\mathcal{D}}{=} W$. □

We now give an example of ssd's constructed with the above scheme. Fix Λ to be a singleton. Then there is no need to specify the marks of the point process $\mathbb{Y}$. Let $\mathbb{Y} = \{\zeta + i, i\in\mathbb{Z}\}$, with $\zeta\sim$ Uniform$[0,1)$. Let X be a white noise process on $\mathbb{R}^n$ with c.f. $C(\phi) = \exp(-(\phi,\phi))$. Then by $I = \Psi^*_{g,H}W$ and (4.5),

$$\langle I,\phi\rangle = \sum_{i=-\infty}^{\infty} \langle B_i, (\Psi_{g,H}\phi)(\zeta+i,\cdot)\rangle, \tag{4.6}$$

with $B_i \stackrel{\mathcal{D}}{=} X$ and $\zeta, B_1, B_2, \ldots$ independent of each other.

Proposition 1. *Fix $k,l\geq 0$, $H\in\mathbb{R}$, and $g\in\mathcal{S}_k(\mathbb{R}^n)$, such that $n+2k>2H$ and $l>-H$. Then I in (4.6) is well-defined and determines an H-ssd on $\mathcal{S}'_l(\mathbb{R}^n)$.*

Proof. First, by Lemma 2, $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_l(\mathbb{R}^n)$ into $L^2(\mathbb{R}\times\mathbb{R}^n)$. Let $\mathcal{F} = \Psi_{g,H}(\mathcal{S}_l(\mathbb{R}^n))$, regarded as a subspace of $L^2(\mathbb{R}\times\mathbb{R}^n)$. To show W is well-defined on $\mathcal{F}$, first, by

$$\begin{aligned}&\int_0^1 \left(\sum_{i=-\infty}^{\infty}\int_{\mathbb{R}^n} |(\Psi_{g,H}\phi)(u+i,v)|^2 dv\right) du \\ &\quad = \int_{\mathbb{R}\times\mathbb{R}^n} |(\Psi_{g,H}\phi)(u,v)|^2\, du\, dv < \infty,\end{aligned}$$

we have

$$S(\zeta) = \sum \int_{\mathbb{R}^n} |(\Psi_{g,H}\phi)(\zeta + i, v)|^2 \, dv < \infty, \quad \text{w.p. } 1$$

Second, whenever $S(u) < \infty$, by the independence of

$$\eta_i = \langle B_i, (\Psi_{g,H}\phi)(u + i, \cdot)\rangle \sim \mathcal{N}(0, \int_{\mathbb{R}^n} |(\Psi_{g,H}\phi)(u + i, v)|^2 \, dv)$$

from each each other, $\sum_i \eta_i$ is a well-defined Gaussian random variable. Therefore W is well-defined.

The c.f. of W is

$$C(f) = E[e^{i\langle W, f\rangle}] = \int_0^1 \exp\left(-\sum_{i=-\infty}^{\infty} \int_{\mathbb{R}^n} |f(u + i, v)|^2 dv\right) du, \quad f \in \mathcal{F}.$$

By $|e^{-x} - e^{-y}| \le |x - y|$, $x, y > 0$, for $f, g \in \mathcal{F}$,

$$\begin{aligned} |C(f) - C(g)| &\le \int_0^1 \left| \sum_{i=-\infty}^{\infty} \int_{\mathbb{R}^n} (|f(u + i, v)|^2 - |g(u + i, v)|^2) \, dv \right| du \\ &\le \|f^2 - g^2\|_1 \le \|f - g\|_2 \|f + g\|_2, \end{aligned}$$

and so C is continuous on $\mathcal{F}$. By Theorem 3, $I = \Psi^*_{g,H} W$, determines an H-ssd on $\mathcal{S}'_l(\mathbb{R}^n)$. □

The "stack-and-project" scheme in Theorem 3 has a useful variant. Loosely speaking, whereas the representation (4.5) "stacks" n-dimensional random fields along the scale domain, the variant stacks $(n + 1)$-dimensional random fields instead.

Corollary 1. *Suppose $\mathcal{F}$ is a linear topological space of functions on $\mathbb{R} \times \mathbb{R}^n$ and is closed under $U_{0,v}$, $v \in \mathbb{R}^n$. Given set Λ, let $\{X_\lambda, \lambda \in \Lambda\}$ be a family of real-valued random functionals on $\mathcal{F}$ that satisfy $X_\lambda \stackrel{\mathcal{D}}{=} U^*_{0,v} X_\lambda$, $v \in \mathbb{R}^n$. Define $\mathbb{Y}$ and $\{Z_i\}$ as in Theorem 3. Suppose for each $f \in \mathcal{F}$,*

$$\langle W, f\rangle = \sum_i \langle Z_i, U_{u_i,0} f\rangle, \tag{4.7}$$

is well-defined w.p. 1 and the c.f. of W is continuous on $\mathcal{F}$. If $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_k(\mathbb{R}^n)$ to $\mathcal{F}$, then $I = \Psi^*_{g,H} W$ is well-defined and determines an H-ssd on $\mathcal{S}'_k(\mathbb{R}^n)$.

Note that in (4.7), $U_{u,0}$ is translation along the scale domain by u. The proof of Corollary 1 follows closely the one for Theorem 3, and hence is omitted.

4.3. Symmetry and self-similarity

Given a test function ϕ, a random generalized function I is called H-ss with respect to ϕ, if $\langle I, \phi\rangle \stackrel{\mathcal{D}}{=} e^{Hu}\langle I, T_v S_{e^u}\phi\rangle$, $u \in \mathbb{R}$, $v \in \mathbb{R}^n$. Denote by E the reflection with respect to 0,

$$E : \phi(x) \to \phi(-x). \tag{4.8}$$

A function ϕ is symmetric (with respect to 0), if $\phi = E\phi$, and anti-symmetric, if $\phi = -E\phi$.

We are interested in the relationship between symmetry and self-similarity. By the observation that symmetric functions and their translated and scaled versions are the "building-blocks" of "good" functions (e.g., in the sense of L^2-theory of wavelets), one may ask if it is true that as long as I is H-ss with respect to all symmetric functions, it is H-ss. Using the "stack-and-project" scheme, we can get a quite complete answer to this question. First, as we will see next, in general, the answer is negative. There are stationary random fields that are H-ss with respect to all symmetric test functions, while not to anti-symmetric ones, and vice versa. Then we will show that under two more conditions which are quite mild, self-similarity with respect to both symmetric functions and anti-symmetric functions guarantees self-similarity. The proofs demonstrate the importance of stationarity along the scale domain in ensuring self-similarity.

Fix $L > 0$. Let N be a Poisson measure on $[0, L] \times \mathbb{R}^n$ with Lebesgue intensity measure. Suppose N_k, $k \in \mathbb{Z}$ are i.i.d.$\sim N$. Then define

$$\langle W_L, f\rangle = \sum_{k=-\infty}^{\infty} (-1)^k \int_{[0,L]\times\mathbb{R}^n} f(u + kL, v) N_k(du,\, dv). \tag{4.9}$$

Note that W_L stacks the stochastic integrals at fixed locations kL, $k \in \mathbb{Z}$. Furthermore, the signs of these integrals change alternatively.

Proposition 2. *(1) Fix $k, l \in \mathbb{N}$, and $H \in \mathbb{R}$, with $k > H$ and $l > -H$. Suppose $g \in \mathcal{S}_k(\mathbb{R}^n)$ is anti-symmetric with respect to 0, i.e. $g = -Eg$. Define $I_L = \Psi^*_{g,H} W_L$. Then for $\phi \in \mathcal{S}_l(\mathbb{R}^n)$ symmetric and $u \in \mathbb{R}$,*

$$\langle I_L, \phi\rangle \stackrel{\mathcal{D}}{=} e^{Hu}\langle I_L, S_{e^u}\phi\rangle. \tag{4.10}$$

(2) On the other hand, for $\mathbb{R}^n = \mathbb{R}$, if $g(x) = \phi(x) = \dfrac{x}{\sqrt{2\pi}} e^{-x^2/2}$ and $H = 0$, then there is $L > 0$, such that $I_L = \Psi^*_{g,H} W_L$ is not H-ss with respect to ϕ. Note that I_L is stationary.

Proof. (1) Because $\phi = E\phi$ and $g = -Eg$, it is not hard to show $(\Psi_{g,H}\phi)(u, v) = -(\Psi_{g,H}\phi)(u, -v)$. Therefore, given $k \geq 1$,

$$\begin{aligned}&\int_{[0,L]\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u + kL, v)\, N_k(du,\, dv)\\ &\quad = -\int_{[0,L]\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u + kL, -v) N_k(du,\, dv), \quad \text{w.p.1}\end{aligned}$$

On the other hand, because $N(\tilde{A}) \stackrel{\mathcal{D}}{=} N(A)$, $A \in \mathcal{B}(\mathbb{R})$, where $\tilde{A} = \{(u, -v) : (u, v) \in A\}$,

$$\int_{[0,L]\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u + kL, -v)\, N_k(du,\, dv) \stackrel{\mathcal{D}}{=} \int_{[0,L]\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u + kL, v) N_k(du,\, dv).$$

The above two formulas, together with the independence among N_k, imply

$$\begin{aligned} \langle I_L, \phi\rangle &= \sum_{k=-\infty}^{\infty} \int_{[0,L]\times\mathbb{R}^n} (-1)^k (\Psi_{g,H}\phi)(u + kL, v)\, N_k(du,\, dv) \\ &\stackrel{\mathcal{D}}{=} \sum_{k=-\infty}^{\infty} \int_{[0,L]\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u + kL, v)\, N_k(du,\, dv) \\ &\stackrel{\mathcal{D}}{=} \int_{\mathbb{R}\times\mathbb{R}^n} (\Psi_{g,H}\phi)(u, v)\, \tilde{N}(du,\, dv), \end{aligned} \tag{4.11}$$

where $\tilde{N}$ is a Poisson measure on $\mathbb{R} \times \mathbb{R}^n$. As will be proved in section 5.2, the right hand side is an H-ssd. Therefore, I_L is H-ss with respect to ϕ.

(2) First of all, since in this case $k = l = 1$ and $H = 0$, by (1), I_L is H-ss with respect to all $\phi \in \mathcal{S}_l(\mathbb{R}^n)$ symmetric. It is easy to check

$$e^{mLH} \langle I_L, S_{e^{mL}}\phi\rangle \stackrel{\mathcal{D}}{=} (-1)^m \langle I_L, \phi\rangle, \quad \phi \in \mathcal{S}_l(\mathbb{R}^n),\ m \in \mathbb{Z}.$$

Then (2) can be proved if for some $L > 0$, $\varphi_L(t) = E[e^{it\langle I_L,\phi\rangle}]$ has non-zero imaginary part. By (4.11),

$$\varphi_L(t) = \exp\left\{ \sum_{k=-\infty}^{\infty} \int_{kL}^{(k+1)L} \int_{\mathbb{R}^n} \left(e^{it(-1)^k \Psi_{g,H}\phi} - 1\right) dv\, du \right\},$$

and hence, letting

$$K(t) = \sum (-1)^k \int_{kL}^{(k+1)L} \int_{\mathbb{R}^n} (\cos((\Psi_{g,H}\phi)(u, v)) - 1)\, dv\, du,$$

we have

$$\begin{aligned} \operatorname{Im} \varphi_L(t) &= e^{K(t)} \sin\left(\sum_{k=-\infty}^{\infty} (-1)^k \int_{kL}^{(k+1)L} \int \sin(t \cdot (\Psi_{g,H}\phi)(u, v))\, dv\, du \right) \\ &= e^{K(t)} \sin(p(t)), \end{aligned}$$

We demonstrate $\operatorname{Im} \varphi_L(t) \not\equiv 0$ by showing $p^{(3)}(0) \neq 0$. First, letting $H = 0$, $\lambda = e^u$, and $\mu = v$,

$$\begin{aligned} (\Psi_{g,H}\phi)(u, v) &= \int \frac{(\lambda x + \mu)x}{2\pi} \exp\left\{ -\frac{(1 + \lambda^2)x^2 + 2\lambda\mu x + \mu^2}{2} \right\} dx \\ &= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2\sigma^2}{2}} (\lambda E\zeta^2 + \mu E\zeta), \end{aligned}$$

with $\sigma^2 = \dfrac{1}{1+\lambda^2}$ and $\zeta \sim \mathcal{N}(-\lambda\mu\sigma^2, \sigma)$. Then $E\zeta^2 = \lambda^2\mu^2\sigma^4 + \sigma^2$, $E\zeta = -\lambda\mu\sigma^2$, and hence

$$\begin{aligned}(\Psi_{g,H}\phi)(u,v) &= \frac{\lambda}{\sqrt{2\pi(1+\lambda^2)^3}} \exp\left\{-\frac{\mu^2}{2(1+\lambda^2)}\right\}\left(1-\frac{\mu^2}{1+\lambda^2}\right)\\ &= \frac{1}{\sqrt{2\pi}} e^u (1+e^{2u})^{-3/2}\left(1-\frac{v^2}{1+e^{2u}}\right)\exp\left\{-\frac{v^2}{2(1+e^{2u})}\right\}.\end{aligned} \tag{4.12}$$

Therefore,

$$\begin{aligned}&p^{(3)}(0)\\ &= \sum_{k=-\infty}^{\infty} (-1)^{(k-1)} \int_{kL}^{(k+1)L} \int ((\Psi_{g,H}\phi)(u,v))^3 \, dv \, du\\ &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} \frac{(-1)^{(k-1)}}{(2\pi)^{3/2}} \frac{1}{\sqrt{3}} \int_{kL}^{(k+1)L} \frac{e^{3u}}{(1+e^{2u})^4} \int \left(1-\frac{\zeta^2}{3}\right)^3 \exp\left\{-\frac{\zeta^2}{2}\right\} d\zeta \, du\\ &= \frac{2}{9\sqrt{3}\pi} \sum_{k=-\infty}^{\infty} (-1)^{(k-1)} \int_{kL}^{(k+1)L} \frac{e^{3u}}{(1+e^{2u})^4} \, du,\end{aligned}$$

where (a) is due to change of variable $v = \sqrt{1+e^{2u}}\zeta/\sqrt{3}$. Then for some $L > 0$, $p^{(3)}(0) \neq 0$. Otherwise, letting $L \to \infty$, there would be

$$\begin{aligned}\int_0^\infty \frac{e^{3u}}{(1+e^{2u})^4} \, du = \int_{-\infty}^0 \frac{e^{3u}}{(1+e^{2u})^4} \, du &= \int_0^\infty \frac{e^{-3u}}{(1+e^{-2u})^4} \, du\\ &= \int_0^\infty \frac{e^{5u}}{(1+e^{2u})^4} \, du,\end{aligned}$$

which is impossible. The second part is thus proved. □

Similarly, we can construct stationary random fields which are H-ss with respect to all ϕ anti-symmetric, while not ϕ symmetric.

Corollary 2. *(1) Fix $k, l \in \mathbb{N}$, and $H \in \mathbb{R}$, with $k > H$ and $l > -H$. Suppose $g \in \mathcal{S}_k(\mathbb{R}^n)$ is symmetric with respect to 0. Define $I_L = \Psi^*_{g,H} W_L$. Then (4.10) holds for all $\phi \in \mathcal{S}_l(\mathbb{R}^n)$ anti-symmetric;*

(2) If $h(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\psi = h''$, and $H = -1$, then there is $L > 0$, such that $I_L = \Psi^*_{h,H} W$ is not H-ss with respect to ψ.

Proof. (1) can be proved as in Proposition 2. For (2), letting $g = \phi = h'$, there is $\Psi_{h,H}\psi = -\Psi_{g,H+1}\phi$, and hence $\langle I_L, \psi\rangle \overset{\mathcal{D}}{=} -\langle \Psi^*_{g,H+1} W_L, \phi\rangle$. As proved in Proposition 2, for some $L > 0$, $\Psi^*_{g,H+1} W_L$ is not $(H+1)$-ss with respect to ϕ. Therefore, I_L is not H-ss with respect to ψ. □

In light of Proposition 2 and Corollary 2, a natural question is that in order for a random field to be H-ss, whether it is enough for the field to be H-ss with respect to all ϕ with $\phi = E\phi$ or $\phi = -E\phi$. With two more assumptions, the answer is affirmative.

Theorem 4. *Suppose I is an ergodic stationary generalized random field on $\mathcal{S}_l(\mathbb{R}^n)$, such that its c.f. C satisfies $|C(\phi)| = |C(E\phi)|$, $\phi \in \mathcal{S}_l(\mathbb{R}^n)$. If I is H-ss with respect to any $\phi \in \mathcal{S}_l(\mathbb{R}^n)$ satisfying either $\phi = E\phi$ or $\phi = -E\phi$, then I is H-ss.*

Proof. Fix $\phi \in \mathcal{S}_l(\mathbb{R}^n)$. Given $\lambda > 0$ and $t \in \mathbb{R}^n$, define $\theta = \lambda^H S_\lambda \phi$ and $h_t(x) = \phi(t/2 + x) + \phi(t/2 - x)$. It is seen $h_t \in \mathcal{S}_l(\mathbb{R}^n)$. Then

$$h_t(x) = h_t(-x), \quad \phi(x) + \phi(t - x) = (T_{-t/2} h_t)(x). \tag{4.13}$$

For $R > 0$,

$$\begin{aligned} E\left[e^{i\langle I,\,\phi\rangle} \times \frac{1}{|B_R|}\int_{B_R} e^{i\langle I,\,T_{-t}E\phi\rangle}\,dt\right] &\overset{(a)}{=} \frac{1}{|B_R|}\int_{B_R} E\left[e^{i\langle I,\,\phi(\cdot)+\phi(t-\cdot)\rangle}\right] dt \\ &\overset{(b)}{=} \frac{1}{|B_R|}\int_{B_R} E\left[e^{i\langle I,\,h_t\rangle}\right] dt. \end{aligned}$$

where (a) is due to Fubini's theorem, (b) to (4.13) and the stationarity of I. Similarly,

$$E\left[e^{i\langle I,\,\theta\rangle} \times \frac{1}{|B_R|}\int_{B_R} e^{i\langle I,\,T_{-t}E\theta\rangle}\,dt\right] = \frac{1}{|B_R|}\int_{B_R} E\left[e^{i\langle I,\,\lambda^H S_\lambda h_t\rangle}\right] dt$$

Because h is symmetric and I is H-ss with respect to h, the right hand sides of the above two formulas are equal, leading to

$$E\left[e^{i\langle I,\,\phi\rangle} \times \frac{1}{|B_R|}\int_{B_R} e^{i\langle I,\,T_{-t}E\phi\rangle}\,dt\right] = E\left[e^{i\langle I,\theta\rangle} \times \frac{1}{|B_R|}\int_{B_R} e^{i\langle I,\,T_{-t}E\theta\rangle}\,dt\right].$$

Let $R \to \infty$. Because I is stationary ergodic, by dominated convergence theorem,

$$C(\phi)C(E\phi) = C(\theta)C(E\theta). \tag{4.14}$$

Similarly, by considering $\phi(t/2 + x) - \phi(t/2 - x) \in \mathcal{S}_l(\mathbb{R}^n)$, we can also get

$$C(\phi)\overline{C(E\phi)} = C(\theta)\overline{C(E\theta)}. \tag{4.15}$$

By $|C(\phi)| = |C(E\phi)|$, $\phi \in \mathcal{S}_l(\mathbb{R}^n)$, (4.14) implies $|C(\phi)| = |C(\theta)|$. If $C(\phi) = 0$, then $C(\theta) = 0$. If $C(\phi) \neq 0$, then by (4.14) and (4.15), $C(\phi)/C(\theta) \in \mathbb{R}$. Therefore, $\arg[C(\phi)] - \arg[C(\theta)] \in \pi\mathbb{Z}$. Since $\theta = \lambda^H S_\lambda \phi$ is continuous in λ, $\arg[C(\phi)] = \arg[C(\theta)]$. Therefore, $C(\phi) = C(\theta) = C(\lambda^H S_\lambda \phi)$, proving I is H-ss. □

5. Infinitely divisible Ssd's

Comparing with the other ssd's, infinitely divisible ssd's can be studied in more detail. For all the representations in this section, we utilize a simple property of U-transform, that is, the transform

$$A_{u_0,v_0} : (u, v) \to (u + u_0, v + e^u v_0)$$

associated with U by $U_{u_0,v_0} f(u, v) = f(A_{u_0,v_0}(u, v))$, is a diffeomorphism with Jacobian 1. Thus, given a random functional $\langle W, f\rangle = \int f(u, v)\, M(du, dv)$, if the random measure M is invariant under any change of variable with Jacobian 1, then

$$\langle W, U_{u_0,v_0} f\rangle = \int f \circ A_{u_0,v_0}\, dM \overset{\mathcal{D}}{=} \int f\, dM = \langle W, f\rangle,$$

and by Theorem 1, $I = \Psi^*_{g,H} W$ determines an H-ssd with the stochastic integral representation

$$\langle I, \phi\rangle = \int (\Psi_{g,H}\phi)(u, v)\, M(du, dv). \tag{5.1}$$

It is relatively easy to find random measures invariant under diffeomorphism with Jacobian 1, such as Poisson measure with Lebesgue intensity measure and stable symmetric random measure with Lebesgue control measure. For spectral representations of infinitely divisible random fields, the above property of U-transform can be similarly utilized.

5.1. *A general spectral representation of infinitely divisible ssd's*

Spectral representations of infinitely divisible random fields were studied in detail in [15]. Given an infinitely divisible random measure Λ on a Euclidean space $\mathcal{E}$ with control measure λ, if $f : \mathcal{E} \to \mathbb{R}$ is measurable and satisfy certain integrability conditions, then it is Λ-integrable, and $\xi = \int f\, d\Lambda$ is an infinitely divisible random variable, with

$$E[e^{i\xi}] = \exp\left\{\int_{\mathcal{E}} \left[ia(s) f(s) - \frac{1}{2} f^2(s)\sigma^2(s) + \int_{\mathbb{R}} \gamma(x, f(s))\, \nu(s, dx)\right] \lambda(ds)\right\}. \tag{5.2}$$

where

$$\gamma(x, t) = e^{ixt} - 1 - \frac{ixt}{1 + x^2}.$$

We now apply (5.2) to random wavelet expansion to get a general result on spectral representation of infinitely divisible ssd's.

Proposition 3. *Assume $\nu(u, v, dx) \equiv \nu(dx)$ and $\lambda(du, dv) = du\,dv$. Suppose the Lévy measure ν satisfies the condition*

$$\int_{|y|\geq 1} |y|\,\nu(dy) < \infty. \tag{5.3}$$

Fix $k, l \geq 0$, $H \in \mathbb{R}$, and $g \in \mathcal{S}_k(\mathbb{R}^n)$, such that $k > H$ and $l > -H$. If $a(s) \equiv a$ and $\sigma^2(s) \equiv \sigma^2$ for constants a and σ, then $\langle I, \phi\rangle = \int \Psi_{g,H}\phi\, d\Lambda$ determines an H-ssd on $\mathcal{S}'_l(\mathbb{R}^n)$, with c.f.

$$C(\phi) = E[e^{i\langle I, \phi\rangle}]$$
$$= \exp\left\{-\frac{\sigma^2}{2}\|\Psi_{g,H}\phi\|_2^2 + \int_{\mathbb{R}\times\mathbb{R}^n}\left[\int_{\mathbb{R}}\gamma(x, (\Psi_{g,H}\phi)(u, v))\,\nu(dx)\right] dudv\right\}. \tag{5.4}$$

Proof. Because $k > H$ and $l > -H$, $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_l(\mathbb{R}^n)$ into $L^1(\mathbb{R}\times\mathbb{R}^n)$ as well as into $L^2(\mathbb{R}\times\mathbb{R}^n)$. Furthermore, either k or l is positive. This implies that for $g \in \mathcal{S}_k(\mathbb{R}^n)$ and $\phi \in \mathcal{S}_l(\mathbb{R}^n)$, $\int \Psi_{g,H}\phi = 0$. By (5.2), $\int a(u, v)(\Psi_{g,H}\phi)(u, v)\,du\,dv = 0$ and hence (5.4) formally holds. It remains to show (5.4) is well-defined.

First, since $k > H$ and $l > -H$, by Lemma 2, for $p = 1, 2$, $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_l(\mathbb{R}^n)$ into $L^p(\mathbb{R}\times\mathbb{R}^n)$, and hence, $\|\Psi_{g,H}\phi\|_p^p$ is finite and continuous in ϕ.

Second, there is $D > 0$, such that $|\gamma(x, t)| \leq Dx^2(|t|+t^2)$, $t \in \mathbb{R}$, $x \in [-1, 1]$. Because for any Lévy measure $\nu(dx)$, $\int_{-1}^1 x^2\nu(dx) < \infty$, therefore,

$$\int_{\mathbb{R}\times\mathbb{R}^n}\int_{-1}^1 \left|\gamma(x,\ (\Psi_{g,H}\phi)(u, v))\right| \nu(dx)\,du\,dv$$
$$\leq D\left(\|\Psi_{g,H}\phi\|_1 + \|\Psi_{g,H}\phi\|_2^2\right)\int_{-1}^1 x^2\,\nu(dx) < \infty,$$

which also shows the first integral is continuous in ϕ.

Third, there is $D' > 0$, such that $|\gamma(x, t)| \leq D'|xt|$, $t \in \mathbb{R}$, $x \in \mathbb{R}\setminus(-1, 1)$. Then by (5.3),

$$\int_{\mathbb{R}\times\mathbb{R}^n}\int_{|x|>1} \left|\gamma(x,\ (\Psi_{g,H}\phi)(u, v))\right| \nu(dx)\,du\,dv$$
$$\leq D'\|\Psi_{g,H}\phi\|_1 \int_{|x|>1} |x|\,\nu(dx) < \infty,$$

which also shows the integral on the left hand side is continuous in ϕ.

Therefore, the integral $\int_{\mathbb{R}\times\mathbb{R}^n}\int_{\mathbb{R}}\gamma(x, (\Psi_{g,H}\phi)(u, v))\,\nu(dx)\,du\,dv$ converges and continuous in ϕ. The proof that the random field determined by (5.4) is H-ss is routine and hence is omitted. □

The proof of Proposition 3 only uses the continuity of $\Psi_{g,H}$ as an operator into L^1 and into L^2. The same idea is used to prove the following variant of (5.4).

Corollary 3. *Suppose $\rho \in L^1(\mathbb{R} \times \mathbb{R}^n) \cap L^2(\mathbb{R} \times \mathbb{R}^n)$. Define a convolution-like transform*

$$K_\rho f(u, v) = \int_{\mathbb{R}\times\mathbb{R}^n} \rho(u - \zeta, v - e^{u-\zeta}\eta) f(\zeta, \eta)\, d\zeta\, d\eta. \tag{5.5}$$

Then, with the same assumptions on ν, k, l, and H as in Proposition 3, the representation

$$\langle I, \phi\rangle = \int (K_\rho \Psi_{g,H}\phi)\, d\Lambda. \tag{5.6}$$

is well-defined and determines an H-ssd on $\mathcal{S}_l'(\mathbb{R}^n)$ with c.f. $C(K_\rho\Psi_{g,H}\phi)$ with C, given by (5.4).

Proof. By the generalized Minkowski inequality, $\|K_\rho f\|_p \le \|f\|_1\|\rho\|_p$, $1 \le p < \infty$. Let $f = \Psi_{g,H}\phi$ and $p = 1, 2$. Since $\Psi_{g,H}$ is a continuous operator from $\mathcal{E} = \mathcal{S}_l(\mathbb{R}^n)$ into $\mathcal{F} = L^1(\mathbb{R} \times \mathbb{R}^n)$, it follows that $K_\rho\Psi_{g,H}$ is a continuous operator from $\mathcal{E}$ into $\mathcal{F}$ as well as into $L^2(\mathbb{R} \times \mathbb{R}^n)$. Then following Proposition 3, $C(K_\rho\Psi_{g,H}\phi)$ is well-defined and continuous in ϕ.

It remains to be shown that $C(K_\rho\Psi_{g,H}\phi)$ is invariant under any U-transform. This can be done by first checking $K_\rho U_{u,v} = U_{u,v}K_\rho$ and then using the invariance of $C(f)$ under U-transforms. □

5.2. Ssd's constructed from Poisson point processes

In the representation (5.1), if M is a Poisson measure with Lebesgue intensity measure, then (5.1) determines a ssd. This observation is part of the following result, where the operator K_ρ defined in (5.5) is applied.

Proposition 4. *Fix $k, l \in \{0\} \cup \mathbb{N}$, $H \in \mathbb{R}$, and $g \in \mathcal{S}_k(\mathbb{R}^n)$, such that $H < k$ and $l > -H$. Also fix $\rho \in L^1(\mathbb{R} \times \mathbb{R}^n)$ such that $\int_{\mathbb{R}\times\mathbb{R}^n} e^{-nu}|\rho(u, v)|\, du\, dv < \infty$. Suppose P is a Poisson point process on $\mathbb{R} \times \mathbb{R}^n$, with Lebesgue intensity measure. Then w.p. 1, for random sample $\{(u_i, v_i)\}$ from P,*

$$\langle I, \phi\rangle = \sum_i (K_\rho\Psi_{g,H}\phi)(u_i, v_i), \tag{5.7}$$

defines $I \in \mathcal{S}_l'(\mathbb{R}^n)$. Moreover, the representation (5.7) determines an H-ss on $\mathcal{S}_l'(\mathbb{R}^n)$ and has finite second moment. The above claims continue to hold if I is defined by

$$\langle I, \phi\rangle = \sum_i (\Psi_{g,H}\phi)(u_i, v_i), \tag{5.8}$$

Proof. We only consider (5.7). The representation (5.8) can be treated in the same way. Let $\eta = |\rho|$. By Lemma 2 (3), in order to show $I \in \mathcal{S}_l'(\mathbb{R}^n)$ w.p. 1, it is enough to get $\sum K_\eta h(u_i, v_i) < \infty$ w.p. 1, which, by Campbell's theorem [10], is implied by $K_\eta h \in L^1$. By the generalized Minkowski inequality, $\|K_\eta h\|_1 \le \|\rho\|_1\|h\|_1 < \infty$.

Similarly, in order to get $E[\langle I, \phi\rangle^2] < \infty$, it is enough that $K_\eta h \in L^2$. Letting $M = \int_{\mathbb{R}\times\mathbb{R}^n} e^{-nu}|\rho(u, v)|\, du\, dv$, the proof proceeds as the following,

$$\begin{aligned}
\|K_\eta h\|_2^2 &= \int_{\mathbb{R}\times\mathbb{R}^n} \left(\int_{\mathbb{R}\times\mathbb{R}^n} |\rho(u-x, v-e^{u-x}y)|h(x, y)\, dx\, dy\right)^2 du\, dv \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} \left(\int_{\mathbb{R}\times\mathbb{R}^n} |\rho(x, v-e^x y)|h(u-x, y)\, dx\, dy\right)^2 du\, dv \\
&\overset{(a)}{\le} \int_{\mathbb{R}\times\mathbb{R}^n} \left(\int_{\mathbb{R}\times\mathbb{R}^n} |\rho(x, v-e^x y)|\, dx\, dy \right. \\
&\qquad \left. \cdot \int_{\mathbb{R}\times\mathbb{R}^n} |\rho(x, v-e^x y)|h^2(u-x, y)\, dx\, dy\right) du\, dv \\
&= M \int_{\mathbb{R}\times\mathbb{R}^n} \left(\int_{\mathbb{R}\times\mathbb{R}^n} |\rho(x, v-e^x y)|h^2(u-x, y)\, dx\, dy\right) du\, dv \\
&= M\|\rho\|_1 \|h\|_2^2 < \infty,
\end{aligned}$$

with (a) due to $\|fg\|_2^2 \le \|f\|_1\|fg^2\|_1$. It is routine to show I is H-ss. We omit the proof for brevity. □

5.3. Stable symmetric ssd's

In representation (5.1), if M is an α-stable symmetric random measure with Lebesgue control measure $du\, dv$, then the representation

$$\langle I, \phi\rangle = \int (\Psi_{g,H}\phi)(u, v)\, M(du, dv), \tag{5.9}$$

defines an α-stable symmetric random field with c.f. [18],

$$E\left[e^{\langle I, \phi\rangle}\right] = \exp\left\{\int_{\mathbb{R}\times\mathbb{R}^n} -|(\Psi_{g,H}\phi)(u, v)|^\alpha\, du\, dv\right\}. \tag{5.10}$$

The conditions for (5.10) to be well-defined and continuous are given in Proposition 5, which immediately follows from Lemma 2 and Theorem 1.

Proposition 5. *Suppose $k, l \ge 0$ and $H \in \mathbb{R}$ satisfy $\alpha(n + k - H) > n$ and $l > -H$. Then for any $g \in \mathcal{S}_k(\mathbb{R}^n)$, (5.9) determines an α-stable symmetric H-ssd on $\mathcal{S}_l'(\mathbb{R}^n)$.*

When $\alpha \in (0, 2)$, the representation (5.10) gives rise to a rich class of ssd's. Two functions in $\mathcal{S}_k(\mathbb{R}^n)$ define the same random field by (5.9) iff they are scaled and translated versions of each other. More specifically, we have

Theorem 5. *Suppose $\alpha \in (0, 2)$ for the stable symmetric random measure M in (5.9). Fix k, l and H as in Proposition 5. Given $g, h \in \mathcal{S}_k(\mathbb{R}^n)$, let I_g and I_h be the random fields determined by g and h (5.9), respectively. Then $I_g \overset{\mathcal{D}}{=} I_h$ if and only if $h = ce^{\beta u_0} S_{e^{u_0}} T_{v_0} g$ for some $(u_0, v_0) \in \mathbb{R}\times\mathbb{R}^n$ and $c = \pm 1$, where $\beta = n(1 - \alpha^{-1}) - H$.*

Proof. The "if" part is easy to verify. To prove the "only if" part, denote $E = \mathbb{R} \times \mathbb{R}^n$. Fix $\phi \in \mathcal{S}_l(\mathbb{R}^n) \setminus \{0\}$ with compact support. For $t_j \in \mathbb{R}$ and $(u_j, v_j) \in E$, $j = 1, \dots, k$, define $\rho = \sum_{i=1}^k t_i e^{Hu_i} T_{v_i} S_{e^{u_i}} \phi$. Then by $\langle I_g, \rho \rangle \overset{\mathcal{D}}{=} \langle I_h, \rho \rangle$, we get

$$\int_E \left| \sum_{i=1}^k t_i (\Psi_{g,H}\phi)(u + u_i, v + e^u v_i) \right|^\alpha du\, dv$$
$$= \int_E \left| \sum_{i=1}^k t_i (\Psi_{h,H}\phi)(u + u_i, v + e^u v_i) \right|^\alpha du\, dv. \tag{5.11}$$

Define binary operation $*$ on E by $(u_1, v_1) * (u_2, v_2) = (u_1 + u_2, e^{u_2} v_1 + v_2)$. Then $(E, *)$ becomes a local compact Hausdorff group with 0 as the identity and $(u, v)^{-1} = (-u, -e^{-u} v)$. The Lebesgue measure is left-invariant under $*$, i.e., for $A \in \mathcal{B}(E)$ and $(u, v) \in E$, $m(A) = m((u, v) * A)$. On the other hand, letting $\Delta(u, v) = e^{-nu}$, there is $\Delta(u, v) m(A * (u, v)) = m(A)$. We can write (5.11) as

$$\int_E \left| \sum_{i=1}^k t_i (\Psi_{g,H}\phi)((u_i, v_i) * (u, v)) \right|^\alpha du\, dv$$
$$= \int_E \left| \sum_{i=1}^k t_i (\Psi_{h,H}\phi)((u_i, v_i) * (u, v)) \right|^\alpha du\, dv.$$

By [9], there is $(\zeta, \eta) \in E$, with $\zeta \in \mathbb{R}$ and $\eta \in \mathbb{R}^n$, and $c = \pm 1$, such that

$$(\Psi_{h,H}\phi)(u, v) = c \left[\Delta((\zeta, \eta)^{-1}) \right]^{1/\alpha} (\Psi_{g,H}\phi)((u, v) * (\zeta, \eta))$$
$$= c e^{n\zeta/\alpha} (\Psi_{g,H}\phi)(u + \zeta, e^\zeta v + \eta).$$

By the duality (3.3), letting $H' = n - H$ and replacing u with $-u$ and v with $-v$,

$$(\Psi_{\phi,H'} h)(u, e^u v) = c e^{n\zeta/\alpha} (\Psi_{\phi,H'} g)(u - \zeta, e^u v - e^{u-\zeta} \eta)$$
$$= c e^{n\zeta/\alpha} (U_{-\zeta, -e^{-\zeta}\eta} \Psi_{\phi,H'} g)(u, e^u v).$$

Writing $u_0 = -\zeta$, $v_0 = -e^{-\zeta}\eta$, and $\beta = -n\alpha^{-1} + H' = n(1 - \alpha^{-1}) - H$, then by (4.2),

$$\Psi_{\phi,H'} h = c e^{-nu_0/\alpha} U_{u_0, v_0} \Psi_{\phi,H'} g = c e^{-nu_0/\alpha} e^{H' u_0} \Psi_{\phi,H'} T_{v_0} S_{e^{u_0}} g$$
$$= c e^{\beta u_0} \Psi_{\phi,H'} T_{v_0} S_{e^{u_0}} g.$$

By Lemma 3, $g \to \Psi_{\phi,H'} g$ is 1-1. Therefore, $h = c e^{\beta u_0} T_{v_0} S_{e^{u_0}} g$. □

In contrast, for $\alpha = 2$, the representation (5.9), which defines a Gaussian ssd with index H, is more redundant. A Gaussian random field on $\mathcal{S}_l(\mathbb{R}^n)$ is H-ss with index $H > 0$ if and only if its c.f. is $C(\phi) = e^{-B(\phi,\phi)}$, with the bilinear functional B given by

$$B(\phi, \psi) = \int_{S^{n-1}} \int_{\mathbb{R}} \hat{\phi}(e^u \omega) \overline{\hat{\psi}(e^u \omega)} e^{2Hu}\, du\, G(d\omega), \tag{5.12}$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $G(E)$ is a finite measure on $\mathcal{B}(S^{n-1})$ such that $G(E) = G(-E)$, for any $E \in \mathcal{B}(S^{n-1})$ [4].

Proposition 6. *Suppose I admits a random wavelet expansion representation (5.9), with $M = B$ an $(n+1)$-dimensional Brownian motion. If the c.f. of I is $e^{-B_g(\phi,\phi)}$, then B_g can be represented by (5.12), with $G(d\omega)$ having a density function*

$$\frac{G(d\omega)}{d\omega} = \bar{G}(\omega) = \int_{\mathbb{R}} c_n e^{(n-2H)u} |\hat{g}(e^u\omega)|^2 \, du = \int_0^\infty c_n t^{n-2H-1} |\hat{g}(t\omega)|^2 dt. \tag{5.13}$$

where c_n is a constant only depending on n. Therefore, if $g, h \in \mathcal{S}_k(\mathbb{R}^n)$ correspond to the same density function $\bar{G}$, then $\int (\Psi_{g,H}\phi)(u, v) B(du, dv) \overset{\mathcal{D}}{=} \int \Psi_{h,H}\phi(u, v)$ $B(du, dv)$.

Proof. By (5.10), $B_g(\phi, \psi) = \langle \Psi_{g,H}\phi, \Psi_{g,H}\psi \rangle$. We need to identify the measure G for B_g. Given u, by $g \in \mathcal{S}_k(\mathbb{R}^n)$, $(\Psi_{g,H}\phi)(u, \cdot) \in \mathcal{S}_k(\mathbb{R}^n)$. Then it is easy to show that

$$(\Psi_{g,H}\phi)^\wedge(u, k) = (2\pi)^{-n/2} \int e^{-ik\cdot v} (\Psi_{g,H}\phi)(u, v) dv = e^{Hu} (2\pi)^{n/2} \hat{g}(k)\hat{\phi}(-e^u k),$$

and likewise for $\hat{\Psi}_{g,H}\psi(u, k)$. Then

$$\begin{aligned} B_g(\phi, \psi) &= \langle (\Psi_{g,H}\phi)^\wedge, (\Psi_{g,H}\psi)^\wedge \rangle \\ &= (2\pi)^n \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{2Hu} |\hat{g}(k)|^2 \hat{\phi}(-e^u k) \overline{\hat{\psi}(-e^u k)} \, dk \, du \\ &\overset{(a)}{=} (2\pi)^n \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{(2H-n)u} |\hat{g}(e^{-u} k)|^2 \hat{\phi}(k) \overline{\hat{\psi}(k)} \, dk \, du, \end{aligned} \tag{5.14}$$

where (a) is due to variable substitution and $|\hat{g}(-e^{-u}k)|^2 = |\hat{g}(e^{-u}k)|^2$. Let $k = e^t\omega$, where $t \in \mathbb{R}$ and $\omega \in S^{n-1}$. Then for some constant c_n, $dk = (2\pi)^{-n} c_n e^{nt} \, dt \, d\omega$, and (5.14) leads to

$$B_g(\phi, \psi) = \int_{\mathbb{R}} \int_{S^{n-1}} \int_{\mathbb{R}} c_n e^{2H(t-u)} e^{nu} |\hat{g}(e^u\omega)|^2 \hat{\phi}(e^t\omega) \overline{\hat{\psi}(e^t\omega)} \, du \, dt \, d\omega.$$

A comparison of the last integral with (5.12) then leads to (5.13). □

6. "Projection-first" perspective of random wavelet expansion

6.1. Operator $L_{g,H}$

Comparing with the image model in section 2, integration over the scale domain is equivalent to projection, whose outcome is a 2D image that contains no direct information on the spatial distribution of objects in the 3D space. Therefore, in order to get meaningful random wavelet expansion by first integrating over the scale domain, the incorporated continuous wavelet expansion should be a function explicitly in the scale as well as in the translate of 2D view of object on the image, rather than the spatial translate of object parallel to the image. This heuristic suggests the following

Definition 3. *Define*

$$(L_{g,H}\phi)(u,v) = (\Psi_{g,H}\phi)(-u, -e^{-u}v). \tag{6.1}$$

By duality (3.3),

$$(L_{g,H}\phi)(u,v) = (\Psi_{\phi,n-H}g)(u,v). \tag{6.2}$$

Define R-transform such that for any $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}^n$, and any f measurable function on $\mathbb{R} \times \mathbb{R}^n$,

$$(R_{u_0,v_0}f)(u,v) = f(u-u_0, e^{-u_0}(v-v_0)). \tag{6.3}$$

Then there is, for any $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}^n$, $(L_{g,H}T_{v_0}S_{e^{u_0}}\phi)(u,v) = e^{-Hu_0}$ $(R_{u_0,v_0}L_{g,H}\phi)(u,v)$.

In analogy to Theorem 1, we have the following results on R-transform and $L_{g,H}$.

Theorem 6. *Suppose that*

(1) $\mathcal{F}$ is a linear topological space of functions on $\mathbb{R} \times \mathbb{R}^n$;
(2) W is a random functional on $\mathcal{F}$ with continuous c.f. C, such that for some β constant, and for all $f \in \mathcal{F}$ and $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}^n$, $C(e^{\beta u_0}R_{u_0,v_0}f) = C(f)$;
(3) $L_{g,H}$ is a continuous operator from $\mathcal{S}_k(\mathbb{R}^n)$ to $\mathcal{F}$.

Then $C \circ L_{g,H}$ is the c.f. of an $(H+\beta)$-ssd on $\mathcal{S}_k'(\mathbb{R}^n)$.

Proof for Theorem 6 is similar to Theorem 1, and hence is omitted. The following result implements the idea of first integrating over the scale domain. A random functional V is called β-ss if it is stationary and $e^{\beta u}S^*_{e^u}V \overset{\mathcal{D}}{=} V$, $u \in \mathbb{R}$.

Theorem 7. *Fix $\mathcal{F}$ and $\mathcal{F}_0$ as linear topological spaces of functions on $\mathbb{R} \times \mathbb{R}^n$ and $\mathbb{R}$ respectively, such that for $f \in \mathcal{F}$ and $v \in \mathbb{R}^n$, $f(\cdot, v) \in \mathcal{F}_0$. Let Λ be a stationary random functional on $\mathcal{F}_0$. Suppose $\mathcal{E}$ is a linear topological space of functions on $\mathbb{R}^n$ and V a β-ss random functional on on $\mathcal{E}$. Assume*

(1) Λ and V are independent;
(2) Given $f \in \mathcal{F}$, with probability 1, $(\Lambda f)(v) \overset{\Delta}{=} \langle \Lambda,\ f(\cdot, v)\rangle$ as a function of v belongs to $\mathcal{E}$;
(3) The c.f. C of $W : f \to \langle V,\ \Lambda f\rangle$ is continuous on $\mathcal{F}$.

If $L_{g,H}$ is continuous from $\mathcal{S}_k(\mathbb{R}^n)$ into $\mathcal{F}$, then $C \circ L_{g,H}$ is the c.f. of an $(H+\beta-n)$-ssd on $\mathcal{S}_k'(\mathbb{R}^n)$ which admits a representation

$$\langle I,\ \phi\rangle = \langle W,\ L_{g,H}\phi\rangle = \langle V,\ \Lambda(L_{g,H}\phi)\rangle. \tag{6.4}$$

Proof. By the continuity of $L_{g,H}$ and C, $C \circ L_{g,H}$ determines a probability distribution on $\mathcal{S}_k'(\mathbb{R}^n)$. It is enough to prove C satisfies Theorem 6 (2). Given $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}^n$, let $f_1(u,v) = f(u, e^{-u_0}(v-v_0))$ and $f_2(u,v) = f(u, e^{-u_0}v)$. Then

$$\langle W,\ R_{u_0,v_0}f\rangle = \langle V,\ \Lambda R_{u_0,v_0}f\rangle \underset{(a)}{\overset{\mathcal{D}}{=}} \langle V,\ \Lambda f_1\rangle \underset{(b)}{\overset{\mathcal{D}}{=}} e^{nu_0}\langle V,\ e^{-nu_0}\Lambda f_2\rangle$$
$$\underset{(c)}{\overset{\mathcal{D}}{=}} e^{(n-\beta)u_0}\langle V,\ \Lambda f\rangle = e^{(n-\beta)u_0}\langle W,\ f\rangle,$$

where (a) is due to stationarity of Λ and independence between Λ and V, (b) to stationarity of V, and (c) to self-similarity with index β of V. $\square$

Based on Theorem 7, we give an example of ssd constructed a Poisson measure and an α-stable symmetric measure.

Proposition 7. *Given $\alpha \in (0, 2]$, fix $k, l \geq 0$, $H \in \mathbb{R}$ such that $\alpha(l + H) > n$ and $n + k - H > 0$. Let N and M be two independent random measures, with N a Poisson measure with Lebesgue intensity measure on $\mathbb{R}$, and M an α-stable symmetric random measure with Lebesgue control measure on $\mathbb{R}^n$. Then given $g \in \mathcal{S}_k(\mathbb{R}^n)$,*

$$\langle I, \phi\rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}} (L_{g,H}\phi)(u, v)\, N(du)\, M(dv), \tag{6.5}$$

determines a $(H - n\alpha^{-1})$-ssd on $\mathcal{S}'_l(\mathbb{R}^n)$.

Proof. Given m large enough, define

$$h(u, v) = \frac{e^{-(H+l)u}}{(1 + |e^{-u} v|)^m} \mathbf{1}_{[0,\infty)}(u) + \frac{e^{(n+k-H)u}}{(1 + |v|)^m} \mathbf{1}_{(-\infty,0)}(u). \tag{6.6}$$

Then by (6.2) and (3.4), $|L_{g,H}\phi| \leq R(\phi)Q(g)h$, with R continuous in ϕ and $R(0) = 0$. By $l + H > 0$, $n + k - H > 0$, and the dominated convergence, this leads to w.p.1, given random sample $\{u_i\}$ from the Poisson point process associated with N, $\int_{\mathbb{R}} (L_{g,H}\phi)(u, v)\, N(du) = \sum_i L_{g,H}\phi(u_i, v)$ is a continuous function for all $\phi \in \mathcal{S}_l(\mathbb{R}^n)$. Write

$$\sum_i h(u_i, v) = \sum_{u_i \geq 0} \frac{e^{-(l+H)u_i}}{(1 + |e^{-u_i} v|)^m} + \sum_{u_i < 0} \frac{e^{(n+k-H)u_i}}{(1 + |v|)^m} = F(v, \{u_i\}) + G(v, \{u_i\}).$$

Then by Theorem 7, it is enough to show that w.p. 1, (1) $F(v, \{u_i\}) \in L^\alpha$, (2) $G(v, \{u_i\}) \in L^\alpha$. Because $n + k - H > 0$, it is easy to prove (2). The proof for (1) is divided into two cases.

Case 1: $\alpha > 1$. By $\alpha(l + H) > n$, fix $\epsilon \in (0, 1)$ such that $n < \alpha(1 - \epsilon)(H + l)$. Let β be such that $\beta^{-1} + \alpha^{-1} = 1$. Then w.p. 1, $S_p = \sum_{u_i \geq 0} e^{\gamma_p u_i} < \infty$ for $p = 1, 2$, with $\gamma_1 = -\beta\epsilon(l + H)$, $\gamma_2 = n - \alpha(1 - \epsilon)(l + H)$. By Hölder's inequality,

$$|F(v, \{u_i\})|^\alpha = \left(\sum_{u_i \geq 0} \frac{e^{-(l+H)u_i}}{(1 + |e^{-u_i} v|)^m} \right)^\alpha \leq S_1^{\alpha/\beta} \sum_{u_i \geq 0} \frac{e^{-\alpha(1-\epsilon)(l+H)u_i}}{(1 + |e^{-u_i} v|)^{m\alpha}}.$$

Integrating both sides gives

$$\begin{aligned} \int_{\mathbb{R}^n} |F(v, \{u_i\})|^\alpha\, dv &\leq S_1^{\alpha/\beta} \int_{\mathbb{R}^n} \sum_{u_i \geq 0} \frac{e^{-\alpha(1-\epsilon)(l+H)u_i}}{(1 + |e^{-u_i} v|)^{m\alpha}}\, dv \\ &\leq S_1^{\alpha/\beta} S_2 \int_{\mathbb{R}^n} \frac{dv}{(1 + |v|)^{m\alpha}} < \infty. \end{aligned}$$

Case 2: $\alpha \le 1$. Fix $R > 0$. Since w.p. 1, $F(v, \{u_i\})$ is a continuous function, it is enough to show, $\int_{|v|\ge R} |F(v, \{u_i\})|^\alpha \, dv < \infty$, w.p.1. By Fubini's theorem, Hölder inequality, and Campbell's theorem,

$$\begin{aligned}
&E\left[\int_{|v|\ge R} |F(v, \{u_i\})|^\alpha \, dv\right] \\
&= \int_{|v|\ge R} E\left(|F(v, \{u_i\})|^\alpha\right) dv \le \int_{|v|\ge R} (E|F(v, \{u_i\})|)^\alpha \, dv \\
&= \int_{|v|\ge R} \left(\int_0^\infty \frac{e^{-(l+H)u}}{(1+|e^{-u}v|)^m} \, du\right)^\alpha dv \\
&\le \int_{|v|\ge R} |v|^{-\alpha(l+H)} \left(\int_{-\infty}^\infty \frac{e^{-(l+H)u}}{(1+e^{-u})^m} \, du\right)^\alpha dv.
\end{aligned}$$

Because $e^{-(l+H)u}(1+|e^{-u}|)^{-m} \le e^{-(l+H)u}\mathbf{1}_{[0,\infty)}(u) + e^{(m-l-H)u}\mathbf{1}_{(-\infty,0)}(u) \in L^1$, and $n < \alpha(l+H)$, the last integral converges. This implies $\int_{|v|\ge R} |F(v, \{u_i\})|^\alpha \, dv < \infty$, w.p. 1.

By (6.6), it is not hard to see the continuity of the c.f. of I. Then by Theorem 7 and the fact that the random functional $f \to \int f \, dM$, $f \in L^\alpha$, is $n(1-\alpha^{-1})$-ss, I is $(H - n\alpha^{-1})$-ss. □

6.2. Construction of linear fractional or log-fractional stable motion by discretization

In this section we demonstrate that the familiar linear fractional stable motion (LFSM) or log-fractional stable motion (log-FSM) defined on $\mathbb{R}$ can be constructed from random wavelet expansion. Let $\mathbb{R}^n = \mathbb{R}$. Suppose M is an α-stable symmetric random measure. Given $k, l \ge 0$, $H \in \mathbb{R}$ satisfying the conditions in Proposition 7, for $g \in \mathcal{S}_k(\mathbb{R})$,

$$\langle I, \phi\rangle = \langle W, L_{g,H}\phi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} (L_{g,H}\phi)(u, v) \, du \, M(dv), \tag{6.7}$$

defines a self-similar generalized random process on $\mathcal{S}'_l(\mathbb{R})$. We will first show that if ϕ is replaced with $\mathbf{1}_{[0,t]}$, the right hand side of (6.7) defines a LFSM. We need the following result.

Lemma 5. *Fix $f \in \mathcal{S}(\mathbb{R})$ and $H > -1$. Define*

$$\tilde{f}(u, v) = f(e^u v) - f(0)\mathbf{1}_{(-\infty,0)}(u)$$

and for $c = \pm 1$,

$$A(c, H, f) = \int_{\mathbb{R}} e^{Hu} \tilde{f}(u, c) \, du, \quad B(c, H, f) = \int_{\mathbb{R}} e^{Hu} |\tilde{f}(u, c)| \, du.$$

Then for any $v \neq 0$,

$$\int_{\mathbb{R}} e^{Hu} \tilde{f}(u,v)\,du = \begin{cases} |v|^{-H}\left(A(\operatorname{sign}(v), H, f) + \dfrac{f(0)}{H}\right) - \dfrac{f(0)}{H}, & H \neq 0, \\ A(\operatorname{sign}(v), 0, f) - f(0)\ln|v|, & H = 0 \end{cases} \tag{6.8}$$

$$\int_{\mathbb{R}} e^{Hu} |\tilde{f}(u,v)|\,du \leq \begin{cases} |v|^{-H}\left(B(\operatorname{sign}(v), H, f) + \dfrac{|f(0)|}{|H|}\right) + \dfrac{|f(0)|}{|H|}, & H \neq 0, \\ B(\operatorname{sign}(v), 0, f) + |f(0)\ln|v||, & H = 0 \end{cases} \tag{6.9}$$

Proof. See section 7. □

Proposition 8. *Fix $H \in (1/\alpha, 1+1/\alpha)$. Given $g \in \mathcal{S}_k(\mathbb{R})$, $k \geq 1$, let $G \in \mathcal{S}_{k-1}(\mathbb{R})$ be such that $G(-\infty) = 0$ and $g = G'$. Then*

$$\{X_t, t \in \mathbb{R}\} = \left\{\int_{\mathbb{R}}\int_{\mathbb{R}} (L_{g,H}\mathbf{1}_{[0,t]})(u,v)\,du\,M(dv), t \in \mathbb{R}\right\} \tag{6.10}$$

can be written as $X_t = \int F(v,t)\,M(dv)$, where for $v \neq 0, t$,

$$F(v,t) = \begin{cases} a(\operatorname{sign}(t-v))|t-v|^{1-H} - a(\operatorname{sign}(-v))|-v|^{1-H}, & H \neq 1 \\ G(0)(\ln|v| - \ln|t-v|) + (a(1) - a(-1)) \\ \quad [\mathbf{1}_{(-\infty,t)}(v) - \mathbf{1}_{(-\infty,0)}(v)], & H = 1 \end{cases}$$

and the constants $a(1)$ and $a(-1)$ are defined by

$$a(c) = \begin{cases} A(c, H-1, G) + G(0)(H-1)^{-1}, & H \neq 1, \\ A(c, 0, G), & H = 1. \end{cases}$$

Remark. When $H \neq 1$, X_t is a LFSM. In [18], a LFSM is represented by $X_t = \int \bar{F}(v,t)\,M(dv)$ with

$$\begin{aligned} \bar{F}(v,t) = a[((t-v)_+)^{\tilde{H}-1/\alpha} - ((-v)_+)^{\tilde{H}-1/\alpha}] \\ + b[((t-v)_-)^{\tilde{H}-1/\alpha} - ((-v)_-)^{\tilde{H}-1/\alpha}], \end{aligned}$$

with $\tilde{H} \in (0,1)$ and $\tilde{H} \neq 1/\alpha$. It is not hard to see that the parameters in F and $\bar{F}$ have the same range. When $H = 1$, (6.10) gives log-fractional stable motion as well as the usual Lévy motion.

Proof. By $H \in (1/\alpha, 1+1/\alpha)$, the process $\int F(v,t)\,M(dv)$ is well-defined. Assume $H \neq 1$. Define $\tilde{G}$ in terms of G as $\tilde{f}$ in terms of f in Lemma 5. By

$$\begin{aligned} (L_{g,H}\mathbf{1}_{[0,t]})(u,v) &= \int_0^t e^{-Hu} g(e^{-u}(x-v))\,dx \\ &= e^{(1-H)u}(G(e^{-u}(t-v)) - G(-e^{-u}v)), \end{aligned}$$

it is seen

$$\int_{\mathbb{R}} (L_{g,H}\mathbf{1}_{[0,t]})(u,v)\,du = \int_{\mathbb{R}} e^{(H-1)u}\tilde{G}(u,t-v)\,du - \int_{\mathbb{R}} e^{(H-1)u}\tilde{G}(u,-v)\,du. \tag{6.11}$$

The case $H \neq 1$ immediately follows from Lemma 5. The case $H = 1$ can be similarly proved. □

We next show the main result of this section, i.e., LFSM can be constructed from random wavelet expansion by discretization. Fix $\rho \in C_0^\infty(-1,1)$ with $\int \rho = 1$. Given $\epsilon > 0$ and t, define

$$\rho_\epsilon(x) = \epsilon^{-1}\rho(\epsilon^{-1}x), \qquad \phi_{\epsilon,t}(s) = \int_{\mathbb{R}} \rho_\epsilon(s)\mathbf{1}_{[0,t]}(x-s)\,ds.$$

It is easy to check $\phi_{\epsilon,t} \in C_0^\infty(I)$, where I is the interval $(\min\{0,t\}-\epsilon, \max\{0,t\}+\epsilon)$. Also, $\phi'_{\epsilon,t}(x) = \operatorname{sign}(t)\,[\rho_\epsilon(x) - \rho_\epsilon(x-t)]$ and $\phi_{\epsilon,t}(x) \to \mathbf{1}_{[0,t]}(x)$.

Theorem 8. *Suppose M is an α-stable symmetric random measure with Lebesgue control measure. Fix $k \geq 1$ and $g \in \mathcal{S}_k(\mathbb{R})$. Let $H \in (1/\alpha, \min\{k, 1+1/\alpha\})$. Define $F_\epsilon(v,t) = \int_{\mathbb{R}} (L_{g,H}\phi_{\epsilon,t})(u,v)\,du$. Then for X in (6.10), $\left\{\int_{\mathbb{R}} F_\epsilon(v,t)\,M(dv)\right\} \xrightarrow{\mathcal{D}} X$, as $\epsilon \to 0$.*

Proof. Denoting by $F(v,t)$ the integral on the left hand side of (6.11), the proof is reduced to showing $\|F_\epsilon(v,t) - F(v,t)\|_\alpha \to 0$, $t \in \mathbb{R}$. Without loss of generality, assume $t > 0$. Then it is easy to see $\phi_\epsilon(x) = 1$, $x \in (\epsilon, t-\epsilon)$. For simplicity, write $\phi_\epsilon = \phi_{\epsilon,t}$ and $\phi = \mathbf{1}_{[0,t]}$. We first prove

$$\lim_{\epsilon\to 0} \int_{\substack{v\notin(-\epsilon,\epsilon)\cup\\(t-\epsilon,t+\epsilon)}} |F_\epsilon(v,t) - F(v,t)|^\alpha\,dv = 0. \tag{6.12}$$

Denote the integral by $I(\epsilon)$. Let $U_\epsilon = (-\epsilon,\epsilon)$, $V_\epsilon = (t-\epsilon, t+\epsilon)$, and $P(u,v) = e^{Hu}g(e^u v)$. Then by

$$F_\epsilon(v,t) - F(v,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{Hu}g(e^u(x-v))(\phi_\epsilon(x) - \phi(x))\,dx\,du,$$

for some constant K,

$$\begin{aligned} I(\epsilon) &\overset{(a)}{\leq} \int_{v\notin U_\epsilon\cup V_\epsilon} \left(\int_{\mathbb{R}}\int_{U_\epsilon\cup V_\epsilon} |P(u,x-v)|\,dx\,du\right)^\alpha dv \\ &\leq K\left[\int_{v\notin U_\epsilon}\left(\int_{U_\epsilon}\int_{\mathbb{R}} |P(u,x-v)|\,du\,dx\right)^\alpha dv \right.\\ &\quad \left. + \int_{v\notin V_\epsilon}\left(\int_{V_\epsilon}\int_{\mathbb{R}} |P(u,x-v)|\,du\,dx\right)^\alpha dv\right], \end{aligned}$$

where (a) is due to $\phi_\epsilon = \phi$ on $\mathbb{R} \setminus (U_\epsilon \cup V_\epsilon)$, and $|\phi_\epsilon - \phi| \le 1$. Since $g \in \mathcal{S}(\mathbb{R})$ and $H > 0$, then

$$\int_{\mathbb{R}} |P(u,v)|\, du = \int_0^\infty \xi^{H-1}|g(v\xi)|\, d\xi \le M|v|^{-H}, \quad v \ne 0$$

where $M = \max\left\{\int_0^\infty \xi^{H-1}|g(c\xi)|\, d\xi, c = \pm 1\right\} < \infty$. Therefore,

$$I(\epsilon) \le KM^\alpha \left[\int_{v\notin U_\epsilon}\left(\int_{U_\epsilon}|x-v|^{-H}\,dx\right)^\alpha dv + \int_{v\notin V_\epsilon}\left(\int_{V_\epsilon}|x-v|^{-H}\,dx\right)^\alpha dv\right].$$

Denote the above two integrals by $I_1(\epsilon)$ and $I_2(\epsilon)$, respectively. By $\alpha H > 1$ and $H < 1 + 1/\alpha$,

$$\begin{aligned} I_1(\epsilon) &= \int_{v\notin U_\epsilon}\left(\int_{U_\epsilon}|x-v|^{-H}\,dx\right)^\alpha dv \\ &= \epsilon^{1+(1-H)\alpha}\int_{|v|\ge 1}\left(\int_{-1}^1 |x-v|^{-H}\,dx\right)^\alpha dv \to 0. \end{aligned}$$

Similarly, $I_2(\epsilon) \to 0$. Therefore, $I(\epsilon) \to 0$ and (6.12) is proved.

For $\tau = 0, t$, as $\epsilon \to 0$, by Proposition 8, $\int_{\tau-\epsilon}^{\tau+\epsilon} |F(v,\tau)|^\alpha\, dv \to 0$. To demonstrate $\|F_\epsilon - F\|_\alpha \to 0$, it remains to be shown that

$$\int_{\tau-\epsilon}^{\tau+\epsilon} |F_\epsilon(v,\tau)|^\alpha\, dv \to 0.$$

First consider the limit for $\tau = 0$. Denote $J(u,v) = L_{g,H}\phi_\epsilon(u,v)$ and let $l = \max\{1, \lfloor H\rfloor\}$. Since $k \ge 1$ and $k+1 > H$, $l \le k$. Fix $f \in \mathcal{S}(\mathbb{R})$, such that $f^{(l)} = g \in \mathcal{S}_k(\mathbb{R})$. Integration by parts gives

$$\begin{aligned} J(u,v) &= \int_{\mathbb{R}} e^{Hu} g(e^u(x-v))\phi_\epsilon(x)\, dx \\ &= (-1)^l \int_{-\epsilon}^{\epsilon} e^{(H-l)u}\rho_\epsilon^{(l-1)}(x)[f(e^u(x-v)) - f(e^u(x+t-v))]\, dx. \end{aligned}$$

Denote

$$j(x,u) = (-1)^l e^{(H-l)u}[f(e^u x) - f(e^u(x+t))].$$

Then $J(u,v) = \int_{-\epsilon}^{\epsilon} \rho_\epsilon^{(l-1)}(x) j(x-v,u)\, dx$. First assume $H \notin \mathbb{N}$. Then $l - H \in (-1,1)\setminus\{0\}$. Then by (6.9), there is $M = M(\epsilon)$, such that

$$\begin{aligned} &\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}} \left|\rho_\epsilon^{(l-1)}(x) j(x-v,u)\right|\, du\, dx \\ &\quad \le M\int_{-\epsilon}^{\epsilon}\left(|x-v|^{l-H} + |x+t-v|^{l-H} + 1\right) dx < \infty. \end{aligned}$$

Therefore, by Fubini's theorem and (6.8),

$$\begin{aligned}\int_{\mathbb{R}} J(u,v)\,du &= \int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}} \rho_\epsilon^{(l-1)}(x)\,j(x-v,u)\,du\,dx\\ &= (-1)^l \int_{-\epsilon}^{\epsilon} \rho_\epsilon^{(l-1)}(x)(a(\mathrm{sign}(x-v))|x-v|^{l-H}\\ &\quad -a(\mathrm{sign}(x+t-v))|x+t-v|^{l-H})\,dx,\end{aligned}$$

where $a(c) = A(c, H-l, f)$, $c=\pm 1$. Notice $\rho_\epsilon^{(l-1)}(x) = \epsilon^{-l}\rho^{(l-1)}(\epsilon^{-1}x)$. Then it is easy to see for some constant K,

$$\begin{aligned}&\int_{-\epsilon}^{\epsilon}\left|\int J(u,v)\,du\right|^\alpha dv\\ &\le \frac{K}{\epsilon^{\alpha l}}\left(\int_{-\epsilon}^{\epsilon}\left(\int_{-\epsilon}^{\epsilon}|x-v|^{l-H}\,dx\right)^\alpha dv+\int_{-\epsilon}^{\epsilon}\left(\int_{-\epsilon}^{\epsilon}|x+t-v|^{l-H}\,dx\right)^\alpha dv\right).\end{aligned}$$

Denote the two integrals on the right hand side by $I_i(\epsilon)$, $i=1,2$, respectively. Then by $l+1>H$, and $H<1+\alpha^{-1}$, as $\epsilon\to 0$,

$$I_1(\epsilon) = \epsilon^{1+\alpha(1-H)}\int_{-1}^{1}\left(\int_{-1}^{1}|x-v|^{l-H}\,dx\right)^\alpha dv \to 0.$$

For $I_2(\epsilon)$, because $x, v\in(-\epsilon,\epsilon)$, when $\epsilon<t/2$, $|x+t-v|$ is lower bounded from 0. Therefore $I_2(\epsilon)\to 0$ as well. Thus, for $\tau=0$,

$$\int_{\tau-\epsilon}^{\tau+\epsilon}|F_\epsilon(v,t)|^\alpha\,dv = \int_{-\epsilon}^{\epsilon}\left|\int_{\mathbb{R}} J(u,v)\,du\right|^\alpha dv\to 0.$$

The cases where $H\in\mathbb{N}$ or $\tau=t$ are similarly proved. Together with (6.12), this proves Theorem 8. □

7. Proofs of technical details

Proof of Lemma 2. We need the following result for the proof.

Lemma 6. *Define two sequences of functionals* $\{K_m,\ m\in\mathbb{N}\}$ *and* $\{b_m,\ m\in\mathbb{N}\}$ *on* $\mathcal{S}(\mathbb{R}^n)$ *by*

$$K_m(f) = \max\left\{\int(1+|x|)^m|f(x)|\,dx,\ \sup_{x\in\mathbb{R}^n}(1+|x|)^m|f(x)|\right\}, \qquad (7.1)$$

$$b_m(f) = 2^m\sup_{x\in\mathbb{R}^n}\left\{(1+|x|)^m\sum_{|\alpha|\le m}|\partial_\alpha f(x)|\right\},\quad f\in\mathcal{S}(\mathbb{R}^n). \qquad (7.2)$$

Suppose $k\ge 0$ *and* $g\in\mathcal{S}_k(\mathbb{R}^n)$. *Then given* $H\in\mathbb{R}$, *for any* $\phi\in\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}|(\Psi_{g,H}\phi)(u,v)| \le \frac{n^k}{k!}\frac{e^{(H-n-k)u}}{(1+|e^{-u}v|)^m}K_{m+k+n+1}(g)b_m(\phi),&\\ m\ge k,\ u\ge 0,\ v\in\mathbb{R}^n.&\qquad (7.3)\end{aligned}$$

Assume Lemma 6 is true for now. We prove the 4 statements in Lemma 2.

(1) To show $\Psi_{g,H}$ is a continuous operator from $\mathcal{S}_l(\mathbb{R}^n)$ into $L^p(\mathbb{R}\times\mathbb{R}^n)$, fix $m>\max\{k,l,n/p\}$. Then $\int_{\mathbb{R}^n}(1+|v|)^{-mp}\,dv<\infty$. By $p(n+k-H)>n$, (7.3) implies

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{R}^n}|\Psi_{g,H}\phi|^p\\
&\quad\le (K_{m+k+n+1}(g)b_m(\phi))^p\left(\frac{n^k}{k!}\right)^p\int_{[0,\infty)\times\mathbb{R}^n}\left(\frac{e^{(H-n-k)u}}{(1+|e^{-u}v|)^m}\right)^p du\,dv\\
&\quad= (K_{m+k+n+1}(g)b_m(\phi))^p\left(\frac{n^k}{k!}\right)^p\int_0^\infty e^{p(H-n-k)u+nu}\,du\int_{\mathbb{R}^n}\frac{dv}{(1+|v|)^{mp}}\\
&\quad<\infty. \qquad (7.4)
\end{aligned}$$

On the other hand, since $\phi\in\mathcal{S}_l(\mathbb{R}^n)$, for $u<0$, by the duality (3.3) and Lemma 6, we have

$$|(\Psi_{g,H}\phi)(u,v)|=|(\Psi_{\phi,n-H}g)(|u|,-e^{|u|}v)|\le\frac{n^l}{l!}\frac{e^{(H+l)u}}{(1+|v|)^m}K_{m+l+n+1}(\phi)b_m(g), \tag{7.5}$$

which, by $l>-H$, leads to

$$\begin{aligned}
&\int_{(-\infty,0)\times\mathbb{R}^n}|\Psi_{g,H}\phi|^p\\
&\le (K_{m+l+n+1}(\phi)b_m(g))^p\left(\frac{n^l}{l!}\right)^p\int_{-\infty}^0 e^{p(H+l)u}\,du\int_{\mathbb{R}^n}\frac{dv}{(1+|v|)^{mp}}<\infty.
\end{aligned} \tag{7.6}$$

Furthermore, by (7.4) and (7.6), for some constant C independent of g and ϕ,

$$\|\Psi_{g,H}\phi\|_p\le C(b_m(g)+K_m(g))(b_m(\phi)+K_m(\phi)), \tag{7.7}$$

It is easy to see that as $\phi\to0$ under the topology of $\mathcal{S}_l(\mathbb{R}^n)$, $b_m(\phi)\to0$ and $K_m(\phi)\to0$. As $\Psi_{g,H}$ is linear, this implies its continuity.

(2) The continuity of $g\to\Psi_{g,H}\phi$ from $\mathcal{S}_k(\mathbb{R}^n)$ into $L^p(\mathbb{R}\times\mathbb{R}^n)$, given $\phi\in\mathcal{S}_l(\mathbb{R}^n)$, is obvious by (7.7).

(3) For any $m\ge\max\{k,l\}$, (7.3) and (7.5) hold. Define

$$R(g)=\max\left\{\frac{n^k}{k!}K_{m+k+n+1}(g),\ \frac{n^l}{l!}b_m(g)\right\},\quad Q(\phi)=\max\{b_m(\phi),\,K_{m+l+n+1}(\phi)\}.$$

Then (3.5) is proved.

(4) The proof of the statement is routine. We therefore omit it for brevity. □

Proof of Lemma 6. Given $u \geq 0$,

$$
\begin{aligned}
&(\Psi_{g,H}\phi)(u,v) \\
&\quad = e^{Hu}\int_{\mathbb{R}^n} g(e^u x + v)\phi(x)\,dx = e^{(H-n)u}\int_{\mathbb{R}^n}\phi(e^{-u}(x-v))g(x)\,dx = I.
\end{aligned}
\tag{7.8}
$$

If $k \geq 1$, then by Taylor's expansion of ϕ around $-e^{-u}v$, and $g \in \mathcal{S}_k(\mathbb{R}^n)$,

$$
\begin{aligned}
I &= e^{(H-n-k)u}\sum_{|\alpha|=k}\int_0^1 \frac{k(1-t)^{k-1}}{\alpha!}\left(\int_{\mathbb{R}^n} x^\alpha \partial_\alpha \phi(te^{-u}x - e^{-u}v)g(x)\,dx\right)dt \\
&= e^{(H-n-k)u}\sum_{|\alpha|=k}\int_0^1 \frac{k(1-t)^{k-1}}{\alpha!}\left(\int_{|x|\leq\frac{1}{2}|v|} F + \int_{|x|\geq\frac{1}{2}|v|} F\right)dt,
\end{aligned}
\tag{7.9}
$$

where F stands for $x^\alpha\partial_\alpha\phi(te^{-u}x - e^{-u}v)g(x)$.

If $|x| \leq \frac{1}{2}|v|$, then for $t \in [0,1]$, $|te^{-u}x - e^{-u}v| \geq \frac{1}{2}e^{-u}|v|$, implying

$$
|\partial_\alpha\phi(te^{-u}x - e^{-u}v)| \leq b_m(\phi)(1+e^{-u}|v|)^{-m}, \quad |\alpha| = k,\ m \geq k, \ldots
$$

with b_m defined by (7.2). Then

$$
\int_{|x|\leq\frac{1}{2}|v|}|F| \leq \frac{b_m(\phi)}{(1+e^{-u}|v|)^m}\int(1+|x|)^k|g(x)|\,dx \leq \frac{b_m(\phi)K_k(g)}{(1+e^{-u}|v|)^m}. \tag{7.10}
$$

If $|x| \geq \frac{1}{2}|v|$, by (7.1) and (7.2),

$$
|x^\alpha g(x)| \leq K_{m+n+k+1}(g)(1+|x|)^{-(m+n+1)}, \quad |\partial_\alpha\phi| \leq 2^{-m}b_m(\phi), |\alpha| = k, m \geq k,
$$

leading to

$$
\int_{|x|\geq\frac{1}{2}|v|}|F| \leq \frac{b_m(\phi)}{2^m}\int_{|x|\geq\frac{1}{2}e^{-u}|v|}\frac{K_{m+n+k+1}(g)}{(1+|x|)^{m+n+1}}\,dx \leq \frac{b_m(\phi)K_{m+n+k+1}(g)}{(1+e^{-u}|v|)^m}.
\tag{7.11}
$$

Because K_m is increasing in m, from (7.9)–(7.11) it is seen that

$$
|(\Psi_{g,H}\phi)(u,v)| \leq \frac{e^{(H-n-k)u}}{(1+e^{-u}|v|)^m}b_m(\phi)K_{m+n+k+1}(g)\sum_{|\alpha|=k}\int_0^1\frac{k(1-t)^{k-1}}{\alpha!}dt.
$$

The summation on the right is equal to $n^k/k!$, completing the proof of (7.3) for $k \geq 1$.

If $k = 0$, then I in (7.8) is decomposed into

$$
I = e^{(H-n)u}\left(\int_{|x|\leq\frac{1}{2}|v|}\phi(e^{-u}(x-v))g(x)\,dx + \int_{|x|\geq\frac{1}{2}|v|}\phi(e^{-u}(x-v))g(x)\,dx\right).
$$

Argument leading to (7.10) and (7.11) still apply, and hence (7.3) is proved. □

Proof of Lemma 3. Define operator $V_{u,v} : \phi(x) \to e^{-nu/2}\phi(e^{-u}x - v)$. Then

$$(\Psi_{g,H}\phi)(u, v) = e^{(H-n/2)u}(\phi, V_{-u,-v}g).$$

Following the proof of (5.14), it is seen that

$$\int_{\mathbb{R}\times\mathbb{R}^n} |(\phi, V_{u,v}g)|^2\, du\, dv = c \int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 \int_{\mathbb{R}} |\hat{g}(e^u\xi)|^2\, du\, d\xi.$$

with $c = (2\pi)^n$. Let $t = e^u|\xi|$. Then $u = \ln t - \ln|\xi|$ and hence

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^n} |(\phi, V_{u,v}g)|^2\, du\, dv &= c \int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 \int_{\mathbb{R}_+} t^{-1}|\hat{g}(t\omega)|^2\, dt\, d\xi \\ &= c \int_{\mathbb{R}^n} |\hat{\phi}(\xi)|^2 c_{g,\omega}\, d\xi. \end{aligned} \tag{7.12}$$

If $(\Psi_{g,H}\phi)(u, v) \equiv 0$, then $(\phi, V_{-u,-v}g) \equiv 0$, and by (7.12), $|\hat{\phi}(\xi)|^2 c_{g,\omega} = 0$. Therefore, for $c_{g,\omega} \neq 0$, $\hat{\phi}(\xi) = 0$. Let $Z = \left\{\omega \in S^{n-1} : c_{g,\omega} = 0\right\}$ and regard it as a topological subspace of S^{n-1}. Denote by Z° the inner part of Z. We prove that as long as $g \neq 0$, $Z^\circ = \emptyset$. First, if $c_{g,\omega} = 0$, then for all $t > 0$, $\hat{g}(t\omega) = 0$. Assume $Z^\circ \neq \emptyset$. Denote by C the cone $\{t\omega : t \in \mathbb{R}_+,\ \omega \in Z^\circ\}$. Then C is open and $\hat{g} \equiv 0$ on C. Given $\alpha, \xi \in C$, $\partial^\alpha \hat{g}(\xi) = \int_{\mathbb{R}^n} (-i)^{|\alpha|} x^\alpha e^{-i\xi\cdot x} g(x)\, dx = 0$. Let $\xi \to 0$. Since 0 is on the boundary of C, and $\hat{g} \in \mathcal{S}(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} x^\alpha g(x)\, dx \equiv 0$. Since g has compact support, then $g = 0$.

The contradiction shows $Z^\circ = \emptyset$. This implies that the set of ξ with $\hat{\phi}(\xi) = 0$ is dense in $\mathbb{R}^n$. Since $\hat{\phi}$ is continuous, this implies $\hat{\phi}(\xi) \equiv 0$, and hence $\phi = 0$. □

Proof of Lemma 5. To prove (6.8), first assume $H \neq 0$. Then for any $v \neq 0$, write $v = c|v|$ and use change of variable to get

$$\begin{aligned}&\int e^{Hu} \tilde{f}(u, v)\, du \\ &= \int_0^\infty e^{Hu} f(e^u v)\, du + \int_{-\infty}^0 e^{Hu}(f(e^u v) - f(0))\, du \\ &= |v|^{-H} \int_{\ln|v|}^\infty e^{Hu} f(ce^u)\, du + |v|^{-H} \int_{-\infty}^{\ln|v|} e^{Hu}(f(ce^u) - f(0))\, du \\ &= |v|^{-H} \left(\int_0^\infty e^{Hu} f(ce^u)\, du + \int_{-\infty}^0 e^{Hu}(f(ce^u) - f(0))\, du \right. \\ &\qquad\qquad \left. - \int_0^{\ln|v|} e^{Hu} f(0)\, du \right) \\ &= |v|^{-H} \left(A(c, H, f) + \frac{f(0)}{H} \right) - \frac{f(0)}{H}. \end{aligned}$$

The case where $H = 0$ is also straightforward. The proof for (6.9) is similar to (6.8) and hence is omitted. □

8. Discussion

In addition to invariance under scaling and translation, we can introduce invariance under other transformations, such as orthogonal transformations, which form a compact group $SO(n)$ and commute with scaling. Indeed, if $C(\phi)$ is the c.f. of an ssd, then $\tilde{C}(\phi) = \int_{SO(n)} C(\phi \circ g) dm(g)$, where m is the Haar measure on $SO(n)$, determines an ssd which is also rotation invariant.

More generally, suppose G is an Abelian Lie group of transformations on $\mathbb{R}^n$ with a finite number of generators $A_1, \dots A_k$. Given $t = (t_1, \dots, t_k)$, let $t \cdot A = \sum_i t_i A_i$. Then elements in G can be represented by $e^{t \cdot A} = \prod e^{t_i A_i}$. Define an operator S_t on test functions ϕ to be $(S_t\phi)(x) = J(e^{t\cdot A})\phi(e^{t\cdot A}x)$, where $J(\cdot)$ is the Jacobian. Let S_t^* be the adjoint of S_t. Formally define a wavelet expansion with multiple index $H = (H_1, \dots, H_k)$ by $(\Psi_{g,H}\phi)(t, v) = e^{t\cdot H} \int (S_t^* T_v^* g) \cdot \phi$. We then get, for any $t_0 \in \mathbb{R}^k$ and $v_0 \in \mathbb{R}^n$, $(\Psi_{g,H} T_{v_0} S_{t_0}\phi)(t, v) = (\Psi_{g,H}\phi)(t + t_0, v + e^{t\cdot A} v_0)$. Introduce operator U_{t_0,v_0} such that $(U_{t_0,v_0} f)(t, v) = f(t + t_0, v + e^{t\cdot A})$. Then we can construct random fields invariant under $U_{t,v}^*$, which induce via $\Psi_{g,H}^*$ random fields invariant under G.

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