stochastic processes and their applications

Stochastic Processes and their Applications 91 (2001) 99–113

www.elsevier.com/locate/spa

Stationary self-similar random fields on the integer lattice

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Received 16 March 1999; received in revised form 11 May 2000; accepted 19 May 2000

Abstract

We establish several methods for constructing stationary self-similar random fields (ssf's) on the integer lattice by "random wavelet expansion", which stands for representation of random fields by sums of randomly scaled and translated functions, or more generally, by composites of random functionals and deterministic wavelet expansion. To construct ssf's on the integer lattice, random wavelet expansion is applied to the indicator functions of unit cubes at integer sites. We demonstrate how to construct Gaussian, symmetric stable, and Poisson ssf's by random wavelet expansion with mother wavelets having compact support or non-compact support. We also generalize ssf's to stationary random fields which are invariant under independent scaling along different coordinate axes. Finally, we investigate the construction of ssf's by combining wavelet expansion and multiple stochastic integrals. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Stationary self-similar; Random wavelet expansion; Multiple stochastic integral; Invariance under independent scaling

1. Introduction

This article establishes several methods to construct stationary and self-similar random fields (ssf's) on the integer lattice $\mathbb{Z}^d$. The methods are based upon "random wavelet expansion", which is motivated by probabilistic image modeling.

Ssf's on $\mathbb{Z}^d$ can be constructed by taking the increments of random fields defined on $\mathbb{R}^d$. For instance, if $Y(t)$ is a self-similar process with stationary increments on $\mathbb{R}$, then $X = \{X_s, s \in \mathbb{Z}\}$, with $X_s = Y(s+1) - Y(s)$, is a ssf on $\mathbb{Z}$ (Samorodnitsky and Taqqu, 1994, Sections 7.2 and 7.10). Other points of views have also been used in the construction of ssf's on $\mathbb{Z}$. Sinai (1976) established a method to construct Gaussian ssf's on $\mathbb{Z}^d$ by studying the bifurcation points of a family of curves defined in a certain space of probability distributions. Dobrushin (1979) studied ssf's on $\mathbb{Z}^d$ by regarding them as discretized ssf's defined on continuum.

Ssf's on the integer lattice constructed in this article are also examples of discretization, which may be understood in the context of image analysis. Given a function f defined on $\mathbb{R}^2$, its digitized image $I = \{I_{ij}, i, j \in \mathbb{Z}\}$ is obtained as the following.

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0304-4149/01/$ - see front matter
PII: S0304-4149(00)00051-X

Fix $d>0$. Divide $\mathbb{R}^2$ into disjoint $d\times d$ squares. Then the value of I_{ij}, termed the "pixel value" of I at (i,j), is given by

$$I_{ij}=\frac{1}{d^2}\int_{id}^{(i+1)d}\int_{jd}^{(j+1)d} f(a,b)\,\mathrm{d}a\,\mathrm{d}b. \tag{1.1}$$

Discretization per se is not related to stationarity and self-similarity of the probability distribution of images. On the other hand, it has been shown that natural images have many empirical scale-invariant properties (Ruderman, 1994; Field, 1994; Mumford and Gidas, 2000). To explain this natural phenomenon, several probabilistic image models have been developed. The term "random wavelet expansion" was coined by Mumford and Gidas (1998), who developed a stationary self-similar image model by considering the composition of the object surfaces in an image. The model proposed by Chi (1998) is from a different point of view. Think of the objects in the 3-D world as planar templates parallel to the image plane. Ignoring the effect of occlusion, an image of the 3-D world is the arithmetic sum of the 2-D views of all the objects, through the lens of a camera. Given an object, assume that when it is located in front of the lens, at distance 1, its 2-D view is described by a function $g(a,b)$ on $\mathbb{R}^2$. Then, within a suitable projective coordinate system, if the object is located at (t,y,z), with t its distance from the camera, then its 2-D view becomes $g(t^{-1}a+y,\,t^{-1}b+z)$. Suppose the locations of all the objects in the 3-D world consist $\{(t_k,y_k,z_k)\}$. For simplicity, assume all the objects look the same. Then an image f of the 3-D world can be written as

$$f(a,b)=\sum_k \text{2-D view of the } k\text{th object}=\sum_k g(t_k^{-1}a+y_k,\,t_k^{-1}b+z_k), \tag{1.2}$$

with g the 2-D view of any of the objects when it is located in front of the lens, at distance 1. Since $\{(t_k,y_k,z_k)\}$ is a random sample from a stochastic point process, f is also random. It can be shown that under certain conditions for the point process, the probability distribution of f is scale and translation invariant (Chi, 2000a; Mumford and Gidas, 2000). Also see Proposition 3.2. Because f is the arithmetic sum of randomly scaled and translated copies of function g, therefore comes the term "random wavelet expansion".

To transform the above image model to random fields defined on $\mathbb{Z}^d$, let I be the digitization of f. Letting $S_{ij}=[id,(i+1)d)\times[jd,(j+1)d)$, by (1.1) and (1.2),

$$I_{ij}=\frac{1}{d^2}\sum_k\int_{S_{ij}} g(t_k^{-1}a+y_k,t_k^{-1}b+z_k)\,\mathrm{d}a\,\mathrm{d}b.$$

Write $t_k=\mathrm{e}^{u_k}$, $v_k=(y_k,z_k)$, $x=(a,b)$. Assume $d=1$ and denote $\phi_{ij}(x)=\mathbf{1}_{[i,i+1)\times[j,j+1)}(x)$. Then $I_{ij}=\sum_k\int g(\mathrm{e}^{u_k}x+v_k)\phi_{ij}(x)\,\mathrm{d}x$. Denoting $\Psi_g\phi(u,v)=\int g(\mathrm{e}^u x+v)\phi(x)\,\mathrm{d}x$ for any function ϕ, we get

$$I_{ij}=\langle W,\Psi_g\phi_{ij}\rangle, \tag{1.3}$$

with $W=\sum_k\delta(u-u_k,v-v_k)$.

Eq. (1.3) indicates two things. First, Ψ_g can be regarded as an expansion of function by scaled and translated copies of g and can be used to build probabilistic image models. With a little abuse of terminology, we call Ψ_g "wavelet expansion with mother wavelet g", although what the term commonly means is a little different from Ψ_g. Despite this, there should be no confusion caused by such usage of the term. Second, random functionals W different from the sum of randomly located δ functions can also be combined with Ψ_g, resulting in different probability distributions on images.

Now we can generalize random wavelet expansion as the composite of a random functional W and the wavelet expansion Ψ_g. The idea of random wavelet expansion has been developed for different function spaces to get different stationary self-similar random fields (Chi, 2000a,b; Chi, 1998; Mumford and Gidas, 2000). In this paper, random wavelet expansion will be defined only for indicator functions of unit squares at integer grid points, i.e., $\mathbf{1}_{[i,i+1)\times[j,j+1)}$.

The so-called "random wavelet expansion" is different from the construction of ssf's by wavelet expansion in its commonly used sense (see, e.g., Meyer et al. (1999) and the references therein). In the latter, wavelets are scaled by factors 2^n, $n \in \mathbb{Z}$, and shifted by $j2^n$, $j \in \mathbb{Z}$, and ssf's arise when the wavelet coefficients are random. In contrast, for random wavelet expansion (1.3), besides the fundamental difference that g is randomly scaled and translated, for all the examples in the paper, the wavelet coefficients are determined by the scales. In order to get ssf's with a given index H (see Definition 1), the coefficient of $g(ta+b)$ is t^H.

In image modeling, it is quite natural to assume an object is bounded. In terms of wavelet expansion, this implies that the mother wavelet g has compact support. In Section 3, random wavelet expansion using such mother wavelets will be used to construct ssf's. On the other hand, random wavelet expansion using mother wavelet with non-compact support can also be applied to construct ssf's, while with more restrictions on the parameters of the random fields. This will be shown in Section 4. With random wavelet expansion, stationary self-similarity can be generalized without extra difficulty. In Section 5, after generalizing wavelet expansion Ψ_g, stationary random fields invariant under "independent scaling" along the coordinate axes will be constructed. All the ssf's constructed in Sections 3–5 can be represented by single stochastic integrals. As is well known, one can construct non-Gaussian ssf's from the Wick powers (multiple Wiener–Itô integrals) of Gaussian ones (Dobrushin, 1979; Taqqu, 1978,1979). To get analogous results for random wavelet expansion, in Section 6, we will investigate how to incorporate it with multiple stochastic integrals. The solution given in this section can be regarded as a generalization of the tensor product of ssf's.

In the next section, we will fix notation. The results on ssf's on the integer lattice will be presented in subsequent sections.

2. Notation

Given $r=(r_1,\ldots,r_d)$, $s=(s_1,\ldots,s_d) \in \mathbb{R}^d$, write $r \leqslant s$ if $r_i \leqslant s_i$, for all $i=1,\ldots,d$. If c is a scalar, then let it also denote the d dimensional vector $(c,\ldots,c)$. Denote the cube $[r_1,s_1)\times\cdots\times[r_d,s_d) \subset \mathbb{R}^d$ by $[r,s)$.

Given random field $X=\{X_s,\ s \in \mathbb{Z}^d\}$ on $\mathbb{Z}^d$, we will always assume that X_s are real-valued. Given $t \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, define translation T_t and scaling S_k such that

$X' = T_t X$ and $X'' = S_k X$ are random fields on $\mathbb{Z}^d$ with

$$X'_s = X_{s+t}, \quad X''_s = \frac{1}{k^d} \sum_{r=ks}^{k(s+1)-1} X_r \tag{2.1}$$

for all $s \in \mathbb{Z}^d$, where $\sum_{r=ks}^{k(s+1)-1}$ stands for summation over all $r = (r_1, \dots, r_d) \in \mathbb{Z}^d$ with $ks_i \leqslant r_i \leqslant k(s_i + 1) - 1$, $i = 1, \dots, d$.

Definition 1. The random field X is called stationary and self-similar with index H (H-ss), if for any $s \in \mathbb{Z}^d$, $X \stackrel{\mathscr{D}}{=} T_s X$, and for any $k \in \mathbb{N}$, $X \stackrel{\mathscr{D}}{=} k^H S_k X$.

Translation and scaling for functions on $\mathbb{R}^d$ are similarly defined. Given function $\phi(x)$ on $\mathbb{R}^d$, for $t \in \mathbb{R}^d$, and $\lambda \in (0, \infty)$, define operators T_t and S_λ such that

$$(T_t \phi)(x) = \phi(x + t) \quad (S_\lambda \phi)(x) = \lambda^{-d} \phi(\lambda^{-1} x).$$

Denote by $\mathscr{S}$ the Schwartz space of infinitely differentiable functions $\phi(x_1, \dots, x_d)$ on $\mathbb{R}^d$ such that for any $m, n_1, \dots, n_d \geqslant 0$,

$$\lim_{|x| \to \infty} (1 + |x|)^m \frac{\partial^{n_1 + \cdots + n_d} f(x)}{\partial^{n_1} x_1 \cdots \partial^{n_d} x_d} = 0.$$

Let $C_0^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions with compact support. Clearly $C_0^\infty(\mathbb{R}^d) \subset \mathscr{S}$. The wavelet g will be chosen from $\mathscr{S}$ or $C_0^\infty(\mathbb{R}^d)$. For more on the space $\mathscr{S}$, see Gel'fand and Vilenkin (1964).

Next, we define wavelet expansion Ψ_g, which is more general than the one introduced in Section 1. The new definition of Ψ_g has an index, allowing us to construct ssf's with different indices.

Definition 2. Given $g \in \mathscr{S}$ with $\int g = 0$, define transformation Ψ_g, such that for any measurable function ϕ on $\mathbb{R}^d$, $\Psi_g \phi$ is a function on $\mathbb{R} \times \mathbb{R}^d$ and

$$(\Psi_g \phi)(u, v) = \int \mathrm{e}^{Hu} g(\mathrm{e}^u x + v) \phi(x) \, \mathrm{d}x = \mathrm{e}^{(H-d)u} \int g(x) (T_v S_{\mathrm{e}^u} \phi)(x) \, \mathrm{d}x, \tag{2.2}$$

where $u \in \mathbb{R}, v \in \mathbb{R}^d$, whenever the integrals in (2.2) are well-defined. We call Ψ_g a wavelet expansion with index H and "mother wavelet" g.

Following the idea of discretization in Section 1, for any $s \in \mathbb{Z}^d$, let $\phi_s = \mathbf{1}_{[s, s+1)}$. From now on we will assume W a random measure. Then (1.3) is rewritten as

$$X_s = \int_{\mathbb{R} \times \mathbb{R}^d} (\Psi_g \phi_s)(u, v) W(\mathrm{d}u, \mathrm{d}v), \quad s \in \mathbb{Z}^d. \tag{2.3}$$

It is easy to check that given $t \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, there are

$$(T_t X)_s = \int (\Psi_g T_t \phi_s) \, \mathrm{d}W, \quad (S_k X)_s = \int (\Psi_g S_k \phi_s) \, \mathrm{d}W. \tag{2.4}$$

For example, for the second identify, letting $X'' = S_k X$, for all $s \in \mathbb{Z}^d$, we have

$$X''_s = \frac{1}{k^d} \sum_{r=ks}^{k(s+1)-1} \int (\Psi_g \phi_r) \, \mathrm{d}W = \frac{1}{k^d} \int (\Psi_g \mathbf{1}_{[ks, k(s+1))}) \, \mathrm{d}W = \int (\Psi_g S_k \phi_s) \, \mathrm{d}W.$$

In the subsequent sections, we will construct ssf's on $\mathbb{Z}^d$ by (2.3). Most of the constructions will be based on stochastic integrals with respect to Poisson and symmetric α-stable processes. Details on such integrals can be found in Kallenberg and Szulga (1989); Major (1981) and Samorodnitsky and Taqqu (1994).

3. Random wavelet expansion using wavelets with compact support

In this section we will construct stationary and self-similar random fields on $\mathbb{Z}^d$ by random wavelet expansion with mother wavelet having compact support. We need some functional properties of $\Psi_g\phi_s$.

Lemma 3.1. *Suppose $g \in C_0^\infty(\mathbb{R}^d)$ and the integral of g along any line parallel to each coordinate axis is 0, i.e., for all $i=1,\dots,d$, and fixed $(x_1,\dots,x_d) \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}} g(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_d)\,\mathrm{d}z = 0. \tag{3.1}$$

Define the wavelet expansion Ψ_g by (2.2). If $H \in (0,d)$, then for $\phi_s = \mathbf{1}_{[s,s+1)}$, with $s \in \mathbb{Z}^d$, and for any $p>0$, $\Psi_g\phi_s \in L^p(\mathbb{R}\times\mathbb{R}^d)$.

The condition (3.1) implies $\int g = 0$ and allows us to construct, for any $H \in (0,d)$, symmetric α-stable or Poisson ssf's with index H. The discussion at the end of the section shows that under (3.1), the high-frequency part of the random field constructed from g is kept limited, which is necessary for the random field to be well defined. The proof of Lemma 3.1 will be given in Section 7. The second lemma, which is straightforward, reveals the relationship between translation, scaling, and wavelet expansion.

Lemma 3.2. *Given $s \in \mathbb{Z}^d$, let $\phi_s = \mathbf{1}_{[s,s+1)}$. Then for any $t \in \mathbb{Z}^d$, and $k \in \mathbb{N}$,*

$$(\Psi_g T_t \phi_s)(u,v) = (\Psi_g\phi_s)(u, v+\mathrm{e}^u t),$$

$$(\Psi_g k^H S_k \phi_s)(u,v) = (\Psi_g\phi_s)(u+\log k, v). \tag{3.2}$$

Based on the results, we can construct various ssf's on $\mathbb{Z}^d$. As an example, we show how to construct symmetric α-stable ssf's in the following. The argument for the construction is standard for all the ssf's in the subsequent sections.

Proposition 3.1. *Let $\alpha \in (0,2]$ and W be a symmetric α-stable random measure on $\mathbb{R}\times\mathbb{R}^d$, with the Lebesgue control measure. Assume the index of Ψ_g is $H \in (0,d)$. Then*

$$X_s = \int_{\mathbb{R}\times\mathbb{R}^d} (\Psi_g\phi_s)(u,v) W(\mathrm{d}u,\mathrm{d}v), \quad s \in \mathbb{Z}^d \tag{3.3}$$

is a well-defined ssf with index H.

Proof. By Lemma 3.1, it is clear that X is well defined. It remains to show that X is a ssf with index H. First, given $s \in \mathbb{Z}^d$, let $\tilde{X} = T_s X$. Then given $\{s_1,\dots,s_N\} \subset \mathbb{Z}^d$, by

(2.4), the characteristic function of $(\tilde{X}_{s_1},\dots,\tilde{X}_{s_N})$ is

$$\exp\left(-\int\left|\sum_{l=1}^{N} x_l\Psi_g\phi_{s_l+s}\right|^\alpha\right) = \exp\left(-\int\left|\sum_{l=1}^{N} x_l\Psi_g T_s\phi_{s_l}\right|^\alpha\right)$$

$$= \exp\left(-\int|\Psi_g T_s f|^\alpha\right),$$

with $f=\sum_{l=1}^N x_l\phi_{s_l}$. By Lemma 3.2, $(\Psi_g T_s f)(u,v)=(\Psi_g f)(u,v+\mathrm{e}^u s)$. Since the transformation $(u,v)\to(u,v+\mathrm{e}^u s)$ has Jacobian 1, $\int|\Psi_g T_s f|^\alpha=\int|\Psi_g f|^\alpha$, which is the characteristic function of $(X_{s_1},\dots,X_{s_N})$. This proves that the distribution of X is stationary. By similar argument, it can be shown that X is self-similar with index H. □

We now construct Poisson ssf's on $\mathbb{Z}^d$. We have the following result.

Proposition 3.2. *Suppose Z is a Poisson point process on $\mathbb{R}\times\mathbb{R}^d$ with intensity measure* $\mathrm{d}u\,\mathrm{d}v$. *If the index of Ψ_g is $H\in(0,d)$, then with probability* 1, *given random sample $\{(u_i,v_i)\}$ from Z,*

$$X_s=\int(\Psi_g\phi_s)(u,v)\,\mathrm{d}Z(u,v)=\sum_i(\Psi_g\phi_s)(u_i,v_i) \tag{3.4}$$

converges absolutely for all $s\in\mathbb{Z}^d$. Furthermore, the random field $X=\{X_s,s\in\mathbb{Z}^d\}$ is a ssf with index H.

Proof. For each $s\in\mathbb{Z}^d$, since $\Psi_g\phi_s\in L^1$, then by Campbell's theorem (Kingman, 1993), with probability 1, $X_s=\sum_i(\Psi_g\phi_s)(u_i,v_i)$ converges absolutely. Because $\mathbb{Z}^d$ is countable, with probability one, X_s is well defined for all $s\in\mathbb{Z}^d$, and hence X is well defined. It is straightforward that X is H-ss. □

We can give the random field in Proposition 3.2 an intuitive explanation. Write

$$X=\left\{\sum_i(\Psi_g\phi_s)(u_i,v_i),\ s\in\mathbb{Z}^d\right\}=\sum_i\left\{(\Psi_g\phi_s)(u_i,v_i),\ s\in\mathbb{Z}^d\right\}=\sum_i \mathrm{e}^{Hu_i}I_i,$$

where each $I_i=\{I_{is},s\in\mathbb{Z}^d\}$ is given by $I_{is}=\int_{[s,s+1)}g(\mathrm{e}^{u_i}x+v_i)\,\mathrm{d}x$, and can be regarded as the digitized image of $g(\mathrm{e}^{u_i}x+v_i)$. Then X is the weighted sum of the images I_i, each being modulated by e^{Hu_i}. Given any $g(\mathrm{e}^{u_i}x+v_i)$, suppose its support is $J_1\times\cdots\times J_d$. Because along any line parallel to any coordinate axis, the integral of g is 0 (Eq. (3.1)), it is not difficult to see that, whenever $J_j\subset[t,t+1)$, for some $j=1,\dots,d$ and $t\in\mathbb{Z}$, there is $\int_{[s,s+1)}g(\mathrm{e}^{u_i}x+v_i)=0$, for any $s\in\mathbb{Z}^d$. This implies that $I_i=0$, or in other words, $g(\mathrm{e}^{u_i}x+v_i)$ is "invisible" in the image X. Therefore, when the support of $g(\mathrm{e}^{u_i}x+v_i)$ is small, the function is visible in X only when it is close to an integer point so that the latter is within the support of the former.

It is clear that terms of the form $g(\mathrm{e}^{u_i}x+v_i)$ with large u_i make up the high-frequency part of the image X. At the same time, these functions have small support. As $u_i\to\infty$, the volume of the support of $g(\mathrm{e}^{u_i}x+v_i)$ decreases like e^{-du_i}. By the above discussion, it is seen that it becomes increasingly unlikely for $g(\mathrm{e}^{u_i}x+v_i)$ to be visible in X.

Together with $H < d$, this leads to the conclusion that the high-frequency part of X is limited. On the other hand, terms of the form $g(e^{u_i}x + v_i)$ having negative u_i with large absolute values make up the low-frequency part of X. Since their images I_i are weighted by e^{Hu_i} with $H > 0$, the contribution to the pixel values of X by I_i decreases exponentially fast as $u_i \to -\infty$. This implies that the low-frequency part of X is also limited. This is consistent with the mathematical conclusion that one can get well-defined Poisson ssf's by random wavelet expansion.

4. Random wavelet expansion using wavelets with non-compact support

The mother wavelet g in Section 3 has compact support and satisfies the condition (3.1) of having vanishing integrals along any line parallel to any coordinate axis. In this section, we will show that wavelets without these two properties can also be used for constructing ssf's on the integer lattice. Understandably, in order to do this, we need more restrictions on the index H.

Lemma 4.1. *Suppose $g \in \mathscr{S}$. For any $s \in \mathbb{Z}^d$, let $\phi_s = \mathbf{1}_{[s,s+1)}$. Given $p \in (1,\infty)$, fix $q > 1$ such that $p^{-1} + q^{-1} = 1$. If $H \in (0, d/q)$, then $\Psi_g \phi_s \in L^p(\mathbb{R} \times \mathbb{R}^d)$.*

Lemma 4.1 will be proved in Section 7. We now can get a result similar to Proposition 3.1 for random wavelet expansion with wavelets not having compact support. The constructed ssf's is symmetric α-stable.

Proposition 4.1. *Given $\alpha \in (1,2]$, let W be a symmetric α-stable random measure on $\mathbb{R} \times \mathbb{R}^d$, with the Lebesgue control measure. Given $g \in \mathscr{S}$, assume the index of Ψ_g is $H \in (0, (1-\alpha^{-1})d)$. Then*

$$X_s = \int (\Psi_g \phi_s)(u,v) W(\mathrm{d}u, \mathrm{d}v), \quad s \in \mathbb{Z}^d$$

is a well-defined ssf with index H.

The proof of Proposition 4.1 is almost identical to Proposition 3.1, hence is omitted. It is worth considering the case $d = 1$ in more detail. We have

Proposition 4.2. *Suppose $d = 1$. Given $g \in \mathscr{S}$, let $G(x)$ be the infinitely differentiable function with $\lim_{x\to-\infty} G(x) = 0$ and $G'(x) = g(x)$. Then for $X = \{X_s, s \in \mathbb{Z}\}$ given in Proposition 4.1, there is $X_s = Y(s+1) - Y(s)$, with*

$$Y = \{Y_t, t \in \mathbb{R}\} = \left\{ \int_{\mathbb{R}\times\mathbb{R}} e^{(H-1)u} [G(e^u t + v) - G(v)] W(\mathrm{d}u, \mathrm{d}v), t \in \mathbb{R} \right\}. \tag{4.1}$$

Furthermore, given $g \in \mathscr{S}$, Y is well defined in the following two cases:

(1) $\alpha \in (1,2]$, *and* $H \in (0, 1-\alpha^{-1})$;
(2) $\int g = 0$, $\alpha \in (0,2]$, *and* $H \in (0,1)$.

Proof. First, from $\int e^{Hu} g(e^u x + v) \mathbf{1}_{[s,s+1)}(x)\, \mathrm{d}x = e^{(H-1)}[G(e^u(s+1)+v) - G(e^u s + v)]$, there is $X_s = Y(s+1) - Y(s)$.

For the remaining part of Proposition 4.2, we only prove part (2). Part (1) can be proved similarly. Without loss of generality, assume $t=1$. Then there is

$$\int_{\mathbb{R}\times\mathbb{R}} e^{(H-1)\alpha u}|G(e^u+v)-G(v)|^\alpha \, du \, dv$$
$$=\int_0^\infty \int_{-\infty}^\infty \xi^{(H-1)\alpha-1}|G(\xi+v)-G(v)|^\alpha \, d\xi \, dv. \tag{4.2}$$

Divide the integral on the right-hand side into two, one on $\{\xi\geqslant 1\}$, the other one on $\{\xi\leqslant 1\}$. For the first integral, since $\int g=0$ implies $G\in\mathscr{S}$, then by first integrating over v and noting $H<1$, it is seen the integral is finite. For the integral on $\{\xi\leqslant 1\}$, given $k\geqslant 1$ with $k\alpha>1$, as $g\in\mathscr{S}$, there is a constant C, such that for all $x\in(0,1)$ and $v\in\mathbb{R}\setminus(-2,2)$, $|g(x+v)|\leqslant C(1+|v|)^{-k}$. Then, letting $h(\xi,v)=\xi^{-1}|G(\xi+v)-G(v)|$, there is

$$h(\xi,v)\leqslant \mathbf{1}_{(-2,2)}(v)\sup\{h(\xi,v)\colon |v|\leqslant 2,\ \xi\in(0,1)\}+\mathbf{1}_{\mathbb{R}\setminus(-2,2)}(v)\frac{1}{\xi}\int_0^\xi |g(x+v)| \, dx$$
$$\leqslant C'\mathbf{1}_{(-2,2)}(v)+C\mathbf{1}_{\mathbb{R}\setminus(-2,2)}(v)\,\frac{1}{(1+|v|)^k},$$

where C' is another constant. Then

$$\int_0^1\int_{-\infty}^\infty \xi^{(H-1)\alpha-1}|G(\xi+v)-G(v)|^\alpha \, d\xi \, dv=\int_0^1\int_{-\infty}^\infty \xi^{H\alpha-1}h^\alpha(\xi,v)\, d\xi \, dv<\infty.$$

Therefore, the integral in (4.2) converges. Hence, Y is well defined. □

From (4.2) we also see that

$$Y\overset{\mathscr{D}}{=}\left\{\int_0^\infty\int_{-\infty}^\infty \xi^{H-1-1/\alpha}[G(\xi t+v)-G(v)]\tilde{M}(d\xi,dv),\ t\in\mathbb{R}\right\},$$

where $\tilde{M}$ is a symmetric α-stable measure with Lebesgue control measure on $\mathbb{R}_+\times\mathbb{R}$. By continuity argument, it is possible to extend the integral to $G=\mathbf{1}_A$, with $A=[-R,R)$. Then for $t>0$,

$$G(\xi t+v)-G(v)=\begin{cases} 1 & -R-\xi t\leqslant v<\min(R-\xi t,-R),\\ -1 & \max(R-\xi t,-R)\leqslant v<R,\\ 0 & \text{otherwise.}\end{cases}$$

Similar equalities hold for $t<0$. In order for the process defined by G to be well defined, it is necessary and sufficient that

$$\int_0^\infty\int_{-\infty}^\infty \xi^{\alpha(H-1)-1}|G(\xi t+v)-G(v)|^\alpha \, d\xi \, dv$$
$$=2\int_0^\infty \xi^{\alpha(H-1)-1}\min(2R,\xi t)\, d\xi<\infty,$$

which holds if and only if $H\in(1-\alpha^{-1},1)$.

5. Stationary random fields on the integer lattice with more than scale invariance

Random wavelet expansion can also be used to construct stationary random fields with self-similarity in a broader sense. For random fields X defined on $\mathbb{R}^d$, such self-similarity would mean for some $h \leqslant d$, there are constants $H_1, \dots, H_h$ and an orthogonal decomposition $\mathbb{R}^d = E_1 \oplus \cdots \oplus E_h$, such that for any $\lambda_1, \dots, \lambda_h > 0$, $X(x) \overset{\mathscr{D}}{=} \lambda_1^{H_1} \cdots \lambda_h^{H_h} X(\lambda_1 P_1 x + \cdots + \lambda_h P_h x)$, with P_j the projection onto E_j.

Before specifying the self-similarity in a broader sense for random fields on $\mathbb{Z}^d$, we introduce the following notation. Given integer $n \geqslant 1$, if $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, and $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n \cup \{\mathbf{0}\}$, then denote $kx = (k_1 x_1, \dots, k_n x_n)$ and $k^x = k_1^{x_1} \cdots k_h^{x_h}$, whenever it is well defined. Note the difference between kx and $k \cdot x = k_1 x_1 + \cdots + k_n x_n$. If f is a function on $\mathbb{R}$, then for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, denote $f(x) = (f(x_1), \dots, f(x_n))$.

Definition 3. Fix $D = (d_1, \dots, d_h) \in \mathbb{N}^h$ such that $\sum_{i=1}^h d_i = d$. For each $r \in \mathbb{Z}^d$, write $r = (r_1, \dots, r_h)$, with $r_i \in \mathbb{Z}^{d_i}$. Given $k = (k_1, \dots, k_h) \in \mathbb{N}^h$, define $S_{D,k}$ to be a scaling transformation with dimensional component D and multiple scaling factor k, such that for any random field X on $\mathbb{Z}^d$, letting $\tilde{X} = S_{D,k} X$, there is

$$\tilde{X}_s = \frac{1}{k^D} \sum_{i=1}^{h} \sum_{r_i = k_i s_i}^{k_i(s_i+1)-1} X_{r_1 \dots r_h},$$

for any $s = (s_1, \dots, s_h)$ with $s_i \in \mathbb{Z}^{d_i}$. A random field X is called stationary self-similar with multiple index $H = (H_1, \dots, H_h)$ and dimensional component $D = (d_1, \dots, d_h)$, or (H, D)-ss for short, if X is stationary and for any $k \in \mathbb{N}^h$, $X \overset{\mathscr{D}}{=} k^H S_{D,k} X$.

Definition 4. With D given as in Definition 3, for each $j = 1, \dots, h$, let P_j denote the map from $\mathbb{R}^d$ into itself, such that for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $P_j x = (\mathbf{0}, x_{d_1 + \cdots + d_{j-1} + 1}, \dots, x_{d_1 + \cdots + d_j}, \mathbf{0})$. Denote $P = (P_1, \dots, P_h)$. As a generalization of wavelet expansion Ψ_g, given multiple index $H = (H_1, \dots, H_h)$, define the transformation $\Psi_{g,H,D}$ such that for any measurable function ϕ, $\Psi_{g,H,D} \phi$ is a function on (u, v), with $u = (u_1, \dots, u_h) \in \mathbb{R}^h$, $v \in \mathbb{R}^d$, and

$$(\Psi_{g,H,D} \phi)(u, v) = \int \mathrm{e}^{u \cdot H} g(\mathrm{e}^{u \cdot P} x + v) \phi(x) \, \mathrm{d}x, \tag{5.1}$$

whenever the integral on the right-hand side is well defined. Recall that $\mathrm{e}^{u \cdot P} x = \sum_{i=1}^h \mathrm{e}^{u_i} P_i x$.

With a little abuse of notation, we also use $P_j x$ to denote $(x_{d_1 + \cdots + d_{j-1} + 1}, \dots, x_{d_1 + \cdots + d_j})$. Parallel to Lemmas 3.1 and 3.2, we have the following properties of $\Psi_{g,H,D}$.

Lemma 5.1. *Fix $g \in C_0^\infty(\mathbb{R}^d)$ such that it satisfies the condition* (3.1). *Given $s \in \mathbb{Z}^d$, let $\phi = \mathbf{1}_{[s,s+1)}$. There is a constant $C = C(g)$, and given $H = (H_1, \dots, H_h)$, if $H_j \in (0, d_j)$, $j = 1, \dots, h$, there are bounded positive functions $f_j(u_j, v_j) = f_j(u_j, v_j; g, H_j, d_j)$ on $\mathbb{R} \times \mathbb{R}^{d_j}$, such that for any $p > 0$, $f_j \in L^p(\mathbb{R} \times \mathbb{R}^{d_j})$, and*

$$|(\Psi_{g,H,D} \phi)(u, v)| \leqslant C \prod_{j=1}^{h} f_j(u_j, P_j v). \tag{5.2}$$

Lemma 5.2. *Let* $\phi_s = \mathbf{1}_{[s,s+1)}$, $\forall s \in \mathbb{Z}^d$. *Then any* $t \in \mathbb{Z}^d$, *and* $k = (k_1, \ldots, k_h) \in \mathbb{N}^h$,

$$(\Psi_{g,H,D} T_t \phi_s)(u, v) = (\Psi_{g,H,D} \phi_s)(u, v + \mathrm{e}^{u \cdot P} t),$$

$$\frac{k^H}{k^D} \sum_{r=ks}^{k(s+1)-1} (\Psi_{g,H,D} \phi_r)(u, v) = (\Psi_{g,H,D} \phi_s)(u + \log k, v). \tag{5.3}$$

Proofs of Lemma 5.1 and Lemma 5.2 will be given in Section 7. From these two lemmas we get the following result, which generalizes Proposition 3.1.

Proposition 5.1. *Given* $\alpha \in (0, 2]$, *suppose* W *is a symmetric* α*-stable random measure on* $\mathbb{R} \times \mathbb{R}^d$ *with the Lebesgue control measure. Assume for* $\Psi_{g,H,D}$ *with* $H = (H_1, \ldots, H_h)$ *and* $D = (d_1, \ldots, d_h)$, *there is* $H_j \in (0, d_j)$, *for any* $j = 1, \ldots, h$. *Then*

$$X_s = \int_{\mathbb{R} \times \mathbb{R}^d} (\Psi_{g,H,D} \phi_s)(u, v) W(\mathrm{d}u, \mathrm{d}v), \quad s \in \mathbb{Z}^d \tag{5.4}$$

is a well-defined ssf with index (H, D).

The proof for the result follows the same line as Proposition 3.1, while using the above two lemmas. We omit the details.

6. Multiple stochastic integral representations of stationary self-similar random fields on the integer lattice

The ssf's on the integer lattice we have so far constructed can be represented by single stochastic integrals. That is, the ssf's with characteristic functions (3.3), (3.4), and (5.4) have the following representation:

$$X_s = \int (\Psi_g \mathbf{1}_{[s,s+1)})(u, v) \, \mathrm{d}Z(u, v), \tag{6.1}$$

where Z is a symmetric α-stable random measure or a Poisson random measure on $\mathbb{R} \times \mathbb{R}^d$, with the Lebesgue control measure. On the other hand, by using multiple Wiener–Itô integrals on the spectral domain, a large class of non-Gaussian ssf's can be constructed (Dobrushin, 1979). The same ssf's can also be constructed using a different type of multiple Wiener–Itô integrals (Taqqu, 1978, 1979). By the analogy between Fourier transform and wavelet expansion, one may ask whether it is possible to combine wavelet expansion with multiple stochastic integrals on the domain of scale and translate, to get ssf's. In this section, this question will be explored.

Lemma 5.1 points out a way to answer the question. In particular, it offers a perspective on the free variable x in the expansion (5.1). That is, each coordinate x_j in $x = (x_1, \ldots, x_d)$ is free, and can be associated with a pair of scale and translate independent of the others. It is possible to impose different stochastic integrals on these independent scales and translates to get ssf's. As can be seen later, this perspective generalizes the idea of tensor products of ssf's. In contrast, in the construction in Dobrushin (1979); Taqqu (1978, 1979), the variable x was taken as a single identity without "inner" freedom. This perspective can be formulated into another way to combine wavelet expansion with multiple stochastic integrals to get ssf's (Chi, 2000b).

We now combine multiple stochastic integrals with results in Section 5. Let $g \in C_0^\infty(\mathbb{R}^d)$ be a function satisfying the condition (3.1). Fix $H=(H_1,\dots,H_h)$ and $D=(d_1,\dots,d_h)$. Then define $\Psi_{g,H,D}$ as in (5.1).

The multiple stochastic integral we will use is defined in Kallenberg and Szulga (1989). Suppose for $j=1,\dots,h$, Z_j is a Poisson random measure or a symmetric α-stable random measure on $\mathbb{R}\times\mathbb{R}^{d_j}$, with the Lebesgue control measure. For simplicity, assume $Z_1,\dots,Z_h$ are independent.

Proposition 6.1. *Suppose for $j=1,\dots,h$, $H_j \in (0,d_j)$. Then, given random measures $Z_1,\dots,Z_h$ as above, such that for each $s\in\mathbb{Z}^d$,*

$$X_s = \int_{\mathbb{R}\times\mathbb{R}^{d_1}} \cdots \int_{\mathbb{R}\times\mathbb{R}^{d_h}} (\Psi_{g,H,D}\mathbf{1}_{[s,s+1)})(u,v)\,\mathrm{d}Z_1(u_1,v_1)\cdots \mathrm{d}Z_h(u_h,v_h), \quad s\in\mathbb{Z}^d, \tag{6.2}$$

is a ssf with index $H_1+\cdots+H_h$.

Proof. That X is well-defined with probability 1 is a direct consequence of Lemma 5.1, the results on multiple Wiener–Itô integrals (Major, 1981), and Theorem 6.2 of Kallenberg and Szulga (1989). It is straightforward to show that X is $(H_1+\cdots+H_h)$-ss. The details of the proof is omitted for simplicity. □

The representation (6.2) is a generalization of tensors of ssf's. Indeed, if $g=g_1\otimes\cdots\otimes g_h$, then X constructed in (6.2) equals $X^{(1)}\otimes\cdots\otimes X^{(h)}$, where $X^{(j)}=\{X_s^{(j)}, s\in\mathbb{Z}^{d_j}\}$ is defined by the single stochastic integral $X_s^{(j)}=\int(\Psi_j\mathbf{1}_{[s,s+1)})(u,v)\,\mathrm{d}Z_j(u,v)$, with Ψ_j an expansion with wavelet g_j and index H_j.

7. Proofs of results on wavelet expansion

In this section we prove the lemmas given in the previous sections. First we prove Lemmas 3.1 and 5.1, which are based on the following results.

Lemma 7.1. *Fix $g\in C_0^\infty[0,\mathrm{e}^{u_0}]$ such that it satisfies the condition* (3.1). *Given $H=(H_1,\dots,H_d)$, with $H_j\in(0,1)$, define*

$$I(u_1,\dots,u_d,v_1,\dots,v_d)=\mathrm{e}^{u\cdot H}\int_{[0,1)} g(\mathrm{e}^{u_1}x_1+v_1,\dots,\mathrm{e}^{u_d}x_d+v_d)\,\mathrm{d}x. \tag{7.1}$$

For each $u\in\mathbb{R}$, let $A_u=[-\mathrm{e}^u,-\mathrm{e}^u+\mathrm{e}^{u_0}]\cup[0,\mathrm{e}^{u_0}]$ and $B_u=[-\mathrm{e}^u,\mathrm{e}^{u_0}]$. Define

$$C=C(g)=\sup(|g|)(1+\mathrm{e}^{u_0})^d. \tag{7.2}$$

Then

$$|I(u_1,\dots,u_d,v_1,\dots,v_d)| \leqslant C\prod_{j=1}^d(\mathbf{1}_{[u_0,\infty)}(u_j)\cdot\mathbf{1}_{A_{u_j}}(v_j)\mathrm{e}^{(H_j-1)u_j}+\mathbf{1}_{(-\infty,u_0)}(u_j)\cdot\mathbf{1}_{B_{u_j}}(v_j)\mathrm{e}^{H_ju_j}). \tag{7.3}$$

In addition, for $u\geqslant u_0$, $m(A_u)=2\mathrm{e}^{u_0}$, and for $u<u_0$, $m(B_u)\leqslant 2\mathrm{e}^{u_0}$.

Proof. Without loss of generality, assume for $j \leqslant k$, $u_j \geqslant u_0$, and for $j > k$, $u_j < u_0$. Write $I = I(u_1,\dots,u_d,v_1,\dots,v_d)$. Letting $J_j = [v_j, \mathrm{e}^{u_j} + v_j] \cap [0,\mathrm{e}^{u_0}]$, by variable substitution,

$$I = \mathrm{e}^{(H_1-1)u_1}\cdots \mathrm{e}^{(H_k-1)u_k}\mathrm{e}^{H_{k+1}u_{k+1}}\cdots \mathrm{e}^{H_d u_d} \times \int_{J_1}\cdots\int_{J_k}\int_{[0,1)}\cdots\int_{[0,1)} g(x_1,\dots,x_k,\mathrm{e}^{u_{k+1}}x_{k+1}+v_{k+1},\dots,\mathrm{e}^{u_d}x_d+v_d)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_d.$$

For $j=1,\dots,k$, if $J_j=\emptyset$, then $I=0$. Moreover, by (3.1), it is seen that if $[v_j,\mathrm{e}^{u_j}+v_j]\supset[0,\mathrm{e}^{u_0}]$, then by integrating with respect to x_j first, there is also $I=0$. Therefore, only when v_j satisfies $[v_j,\mathrm{e}^{u_j}+v_j]\cap[0,\mathrm{e}^{u_0}]\neq\emptyset$ and $[v_j,\mathrm{e}^{u_j}+v_j]\not\supset[0,\mathrm{e}^{u_0}]$, which implies $v_j\in A_{u_j}$, can I be non-zero.

On the other hand, for $j=k+1,\dots,d$ and $x\in[0,1)$, $\mathrm{e}^{u_j}x+v_j\in[v_j,\mathrm{e}^{u_j}+v_j]$. Since the support of g is in $[0,\mathrm{e}^{u_0}]^d$, only when $[v_j,\mathrm{e}^{u_j}+v_j]\cap[0,\mathrm{e}^{u_0}]\neq\emptyset$, which implies $v_j\in B_{u_j}$, can I be non-zero. Therefore, we get

$$|I|\leqslant|I|\prod_{j=1}^{d}(\mathbf{1}_{[u_0,\infty)}(u_j)\cdot\mathbf{1}_{A_{u_j}}(v_j)+\mathbf{1}_{(-\infty,u_0)}(u_j)\cdot\mathbf{1}_{B_{u_j}}(v_j)). \tag{7.4}$$

For u and v with $I\neq 0$, since the region of integral I is contained in $R=[0,\mathrm{e}^{u_0}]^k\times[0,1)^{d-k}$,

$$\begin{aligned}|I| &\leqslant \sup(|g|)\prod_{j=1}^{k}\mathrm{e}^{(H_j-1)u_j}\prod_{j=k+1}^{d}\mathrm{e}^{H_ju_j}m(R)\\ &\leqslant \sup(|g|)(1+\mathrm{e}^{u_0})^d\prod_{j=1}^{k}\mathrm{e}^{(H_j-1)u_j}\prod_{j=k+1}^{d}\mathrm{e}^{H_ju_j}.\end{aligned}$$

This together with (7.4) and (7.2) proves the lemma. □

Proof of Lemma 3.1. Without loss of generality, assume the support of g is in $[0,\mathrm{e}^{u_0}]$. Given index $H\in(0,d)$ of the wavelet expansion Ψ_g and $\phi=\mathbf{1}_{[0,1)}$, let $H_j=H/d$, for $j=1,\dots,d$. Then from (7.1) and (7.3), for any $u\in\mathbb{R}$, $v\in\mathbb{R}^d$,

$$\begin{aligned}|(\Psi_g\phi)(u,v)| &= |I(u,\dots,u,v_1,\dots,v_d)|\\ &\leqslant C\prod_{j=1}^{d}(\mathbf{1}_{[u_0,\infty)}(u)\cdot\mathbf{1}_{A_u}(v_j)\mathrm{e}^{(H/d-1)u}+\mathbf{1}_{(-\infty,u_0)}(u)\cdot\mathbf{1}_{B_u}(v_j)\mathrm{e}^{Hu/d})\\ &= C(\mathbf{1}_{[u_0,\infty)}(u)\mathbf{1}_{A_u^d}(v)\mathrm{e}^{(H-d)u}+\mathbf{1}_{(-\infty,u_0)}(u)\cdot\mathbf{1}_{B_u^d}(v)\mathrm{e}^{Hu}).\end{aligned} \tag{7.5}$$

Then it is easy to see that $\Psi_g\phi\in L^\infty(\mathbb{R}\times\mathbb{R}^d)$. For the L^p bound on $|(\Psi_g\phi)(u,v)\mathbf{1}_{[u_0,\infty)}(u)|$, for any $0<p<\infty$, since $H\in(0,d)$ and $m(A_u^d)=(2\mathrm{e}^{u_0})^d$ is a constant,

$$\begin{aligned}\int|(\Psi_g\phi)(u,v)\mathbf{1}_{[u_0,\infty)}(u)|^p\,\mathrm{d}u\,\mathrm{d}v &\leqslant C^p\int_{u_0}^{\infty}\mathrm{d}u\int_{A_u^d}\mathrm{e}^{p(H-d)u}\,\mathrm{d}v\\ &\leqslant C^p(2\mathrm{e}^{u_0})^d\int_{u_0}^{\infty}\mathrm{e}^{p(H-d)u}\,\mathrm{d}u<\infty\end{aligned}$$

For the L^p bound on $(\Psi_g\phi)(u,v)\mathbf{1}_{(-\infty,u_0]}(u)$, for any $0<p<\infty$, since $H\in(0,d)$ and $m(B_u^d)\leqslant 2^d e^{du_0}$, then

$$\int |(\Psi_g\phi)(u,v)\mathbf{1}_{(-\infty,u_0]}(u)|^p\,du\,dv \leqslant C^p\int_{-\infty}^{u_0} du\int_{B_u^d} e^{pHu}\,dv$$
$$\leqslant C^p(2e^{u_0})^d\int_{-\infty}^{u_0} e^{pHu}\,du<\infty.$$

Therefore $(\Psi_g\phi)(u,v)\in L^p$. For $\phi=\mathbf{1}_{[s,s+1)}$ with arbitrary $s\in\mathbb{Z}^d$, the result is similarly proved. □

With more complex notation, Lemma 5.1 can be similarly proved. The key step is again to get from (7.3) a bound on $|(\Psi_{g,H,D}\phi)(u,v)|$ similar to (7.5). For simplicity, the details of the proof is omitted.

Proof of Lemma 4.1. Let $N=2\sqrt{d}$. Without loss of generality, we will only consider $\phi=\mathbf{1}_{[0,1)}$. To show the function is L^p integrable, consider $(\Psi_g\phi)(u,v)$ on 3 regions, $R_1=\{(u,v)\colon u\geqslant 0, |v|\geqslant Ne^u\}$, $R_2=\{(u,v)\colon u\geqslant 0, |v|<Ne^u\}$, and $R_3=\{(u,v)\colon u<0\}$.

For the region R_1, by variable substitution,

$$(\Psi_g\phi)(u,v)=e^{Hu}\int_{[0,1)} g(e^u x+v)\,dx=e^{(H-d)u}\int_{[v,v+e^u\cdot 1)} g(x)\,dx. \tag{7.6}$$

Since $g\in\mathscr{S}$, there is a constant $C_1>0$, such that $|g(x)|\leqslant C_1(1+|x|)^{-2d-2}$, for all $x\in\mathbb{R}^d$. Then $|(\Psi_g\phi)(u,v)|\leqslant e^{(H-d)u}\int_{[v,v+e^u\cdot 1)} C_1(1+|x|)^{-2d-2}\,dx$. Given any $x\in[v,v+e^u\cdot 1)$, since $|v|\geqslant Ne^u$ for $(u,v)\in R_1$, there is $|x|\geqslant|v|-|v-x|\geqslant|v|-|e^u\cdot 1|=|v|-\sqrt{d}e^u\geqslant\frac{1}{2}|v|$. Therefore, we get, for some constant C_2,

$$|(\Psi_g\phi)(u,v)|\leqslant e^{(H-d)u}\int_{|x|\geqslant\frac{1}{2}|v|}\frac{C_1}{(1+|x|)^{2d+2}}\,dx\leqslant\frac{C_2e^{(H-d)u}}{(1+|v|)^{d+1}},\quad (u,v)\in R_1.$$

Since $H<(1-p^{-1})d<d$, it is then clear the integral of $|(\Psi_g\phi)(u,v)|^p$ over R_1 is finite.

For the second region R_2, from (7.6) we get $|(\Psi_g\phi)(u,v)|\leqslant e^{(H-d)u}\int|g|$. Therefore, for a constant C_3,

$$\int_{R_2}|(\Psi_g\phi)(u,v)|^p\,du\,dv\leqslant C_3\int_{\substack{u\geqslant 0\\ |v|<Ne^u}} e^{p(H-d)u}\,du\,dv$$
$$\leqslant C_3\int_{u\geqslant 0} e^{p(H-d)u}N^d e^{du}\,du.$$

From $p(H-d)+d=p(H-(1-p^{-1})d)<0$, we get that the integral over R_2 is finite.

Finally, for $(u,v)\in R_3$, since $u<0$, $e^u<1$. Therefore, for any $x\in[0,1)$, $|e^u x+v|\leqslant|e^u\cdot 1|+|v|\leqslant\sqrt{d}+|v|$. By $g\in\mathscr{S}$, it is then seen for some constant C_4, there is $|g(e^u x+v)|\leqslant C_4(1+|v|)^{-d-1}$, for all $x\in[0,1)$, $v\in\mathbb{R}^d$. By (7.6), this implies that

$$|(\Psi_g\phi)(u,v)|\leqslant e^{Hu}\int_{[0,1)}\frac{C_4}{(1+|v|)^{d+1}}\,dx=\frac{C_4e^{Hu}}{(1+|v|)^{d+1}}.$$

Integrate $|(\Psi_g\phi)(u,v)|^p$ over the region $\{u<0\}\times\mathbb{R}^d$. Since $H>0$, then the integral on R_3 is also finite. This completes the proof that $\Psi_g\phi\in L^p(\mathbb{R}\times\mathbb{R}^d)$. □

Proof of Lemma 5.2. For the first equation in (5.3), for any $t\in\mathbb{Z}^d$,

$$\begin{aligned}(\Psi_{g,H,D}T_t\phi_s)(u,v) &= \int e^{u\cdot H} g(e^{u\cdot P}x+v)T_t\phi_s(x)\,dx\\ &= \int e^{u\cdot H} g(e^{u\cdot P}x+v)\phi_s(x-t)\,dx\\ &= \int e^{u\cdot H} g(e^{u\cdot P}(x+t)+v)\phi_s(x)\,dx\\ &= (\Psi_{g,H,D}\phi_s)(u,v+e^{u\cdot P}t).\end{aligned}$$

For the second equation in (5.3), for any $k=(k_1,\dots,k_h)\in\mathbb{N}^d$, $s=(s_1,\dots,s_h)\in\mathbb{Z}^d$, and $x=(x_1,\dots,x_h)\in\mathbb{R}^d$.

$$\begin{aligned}\tilde\phi_s(x) &= \frac{1}{k^D}\sum_{\substack{k_is_i\leqslant r_i\leqslant\\ k_i(s_i+1)-1}}\phi_{r_1\dots r_h}(x) = \frac{1}{k^D}\sum_{\substack{k_is_i\leqslant r_i\leqslant\\ k_i(s_i+1)-1}}\prod_{i=1}^h \mathbf{1}_{[r_i,r_i+1)}(x_i)\\ &= \frac{1}{k^D}\prod_{i=1}^h \mathbf{1}_{[k_is_i,k_i(s_i+1))}(x_i) = \frac{1}{k^D}\prod_{i=1}^h \mathbf{1}_{[s_i,s_i+1)}\left(\frac{x_i}{k_i}\right) = \frac{1}{e^{\log k\cdot D}}\phi_s(e^{-\log k\cdot P}x).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{k^H}{k^D}\left(\Psi_{g,H,D}\sum_{k_is_i\leqslant r_i\leqslant(k_i+1)s_i-1}\phi_{r_1\dots r_h}\right)(u,v)\\ &\quad= \int k^H e^{u\cdot H} g(e^{u\cdot P}x+v)\tilde\phi_s(x)\,dx\\ &\quad= \int e^{(\log k)\cdot H} e^{u\cdot H} g(e^{u\cdot P}x+v)e^{-\log k\cdot D}\phi_s(e^{-(\log k)\cdot P}x)\,dx\\ &\quad= \int e^{(\log k+u)\cdot H} g(e^{(\log k+u)\cdot P}x+v)\phi_s(x)\,dx = (\Psi_{g,H,D}\phi_s)(u+\log k,v).\end{aligned}$$

This completes the proof of Lemma 5.2. □

Acknowledgements

Part of the paper comes from the author's PhD dissertation at Brown University under Professor Stuart Geman.

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