# BAYESIAN INFERENCE OF HIDDEN GAMMA WEAR PROCESS MODEL FOR SURVIVAL DATA WITH TIES

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This supplementary material has two sections. Section S1 contains proofs of the theoretical results of the paper. Section S2 is a discussion on possible extension to wear processes other than the Gamma processes considered in the paper.

## S1. Proofs

**Proof of Proposition 3.1.** Given $i \neq j$, let $S(t) = P(T_i > t_1, T_j > t_2)$ for $t = (t_1, t_2)$. Then $S(t) = E[P(T_i > t_1, T_j > t_2 \,|\, \mathcal{W})] = E[e^{-H_i(t_1)-H_j(t_2)}]$. If $t_1 \le t_2$, then by $H_i(t_1) + H_j(t_2) = H_i(t_1) + H_j(t_1) + [H_j(t_2) - H_j(t_1)]$ and independence between $\{H(s), s \le t_1\}$ and $\{H(s) - H(t_1), s > t_1\}$, $S(t) = e^{-\Psi(\kappa_{\{i,j\}},t_1)-[\Psi(\kappa_j,t_2)-\Psi(\kappa_j,t_1)]}$. Since $\ln S(t)$ can be written as $f(t_1) + g(t_2)$ for two univariate functions $f$ and $g$, $D_1 D_2[\ln S(t)] = 0$, giving $\theta^*_{T_i T_j}(t) = 1$. The case $t_1 \ge t_2$ can be shown likewise. □

**Proof of Theorem 3.2.** With a little abuse of notation, denote $H_i(t, u) = H_i(u) - H_i(t)$ and $\Psi(a, t, u) = \Psi(a, u) - \Psi(a, t)$ for $0 \le t \le u$. Let

$$\psi(a,t) = \int_0^\infty (1 - e^{-a's})\, \varphi(\mathrm{d}s \,|\, t). \tag{S1.1}$$

Then $\partial\Psi(a,t)/\partial t = \psi(a,t)$ for almost every $t$. Given data $D_{\mathrm{obs}}$, fix $\epsilon > 0$, such that $\tau_{j-1} < \tau_j - \epsilon$ for $j \ge 1$. Then

$$\begin{aligned}
&P\left(y_i - \epsilon < T_i \le y_i \text{ for } i \in \mathscr{D} \text{ and } T_i > y_i \text{ for } i \in \mathscr{N} \,|\, \mathcal{W}\right) \\
&= \prod_{j=1}^N \left\{ \prod_{i \in \mathscr{D}_j} \left[ e^{-H_i(\tau_j - \epsilon)} - e^{-H_i(\tau_j)} \right] \prod_{i \in \mathscr{N}_j} e^{-H_i(\tau_j)} \right\} \\
&= \prod_{j=1}^N \left\{ \prod_{i \in \mathscr{D}_j} e^{-H_i(\tau_j - \epsilon)} [1 - e^{-H_i(\tau_j - \epsilon, \tau_j)}] \prod_{i \in \mathscr{N}_j} e^{-H_i(\tau_j)} \right\}.
\end{aligned}$$

By $H_i(\tau_j-\epsilon)=\sum_{l=1}^{j}H_i(\tau_{l-1},\tau_l-\epsilon)+\sum_{l=1}^{j-1}H_i(\tau_l-\epsilon,\tau_l)$ and a similar decomposition of $H_i(\tau_j)$, the right hand side is equal to

$$\prod_{j=1}^{N}\left\{\prod_{i\in\mathscr{D}_j}e^{-\sum_{l=1}^{j}H_i(\tau_{l-1},\tau_l-\epsilon)-\sum_{l=1}^{j-1}H_i(\tau_l-\epsilon,\tau_l)}\right.$$
$$\left.\times\prod_{i\in\mathscr{N}_j}e^{-\sum_{l=1}^{j}H_i(\tau_{l-1},\tau_l-\epsilon)-\sum_{l=1}^{j}H_i(\tau_l-\epsilon,\tau_l)}\times\prod_{i\in\mathscr{D}_j}\left[1-e^{-H_i(\tau_j-\epsilon,\tau_j)}\right]\right\},$$

which, after rearranging the terms and changing indexes, is equal to

$$\prod_{j=1}^{N}\left\{e^{-\sum_{i\in\mathscr{R}_j}H_i(\tau_{j-1},\tau_j-\epsilon)}e^{-\sum_{i\in\mathscr{R}'_j}H_i(\tau_j-\epsilon,\tau_j)}\prod_{i\in\mathscr{D}_j}\left[1-e^{-H_i(\tau_j-\epsilon,\tau_j)}\right]\right\}.$$

Since $\mathcal{W}=(H_1,\ldots,H_n)$ has independent increments, taking expectation with respect to its law yields

$$P\left(y_i-\epsilon<T_i\le y_i\text{ for }i\in\mathscr{D}\text{ and }T_i>y_i\text{ for }i\in\mathscr{N}\right)=\prod_{j=1}^{N}\phi_{1j}(\epsilon)\prod_{j=1}^{N}\phi_{2j}(\epsilon),\tag{S1.2}$$

where for each $j\le N$, $\phi_{1j}(\epsilon)=e^{-\Psi(\varrho_j,\tau_{j-1},\tau_j-\epsilon)}$ and

$$\phi_{2j}(\epsilon)=E\left\{e^{-\sum_{i\in\mathscr{R}'_j}H_i(\tau_j-\epsilon,\tau_j)}\prod_{i\in\mathscr{D}_j}[1-e^{-H_i(\tau_j-\epsilon,\tau_j)}]\right\}.$$

As $\epsilon\to 0$, $\phi_{1j}(\epsilon)\to\phi_{1j}(0)=e^{-\Psi(\varrho_j,\tau_{j-1},\tau_j)}$. On the other hand, expansion of $\prod_{i\in\mathscr{D}_j}[1-e^{-H_i(\tau_j-\epsilon,\tau_j)}]$ gives

$$\begin{aligned}\phi_{2j}(\epsilon)&=E\left[\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}\,e^{-\sum_{i\in\mathscr{R}'_j\cup T}H_i(\tau_j-\epsilon,\tau_j)}\right]\\&=\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}\,e^{-\Psi(\omega_j+\kappa_T,\,\tau_j-\epsilon,\,\tau_j)}.\end{aligned}$$

If $\mathscr{D}_j=\emptyset$, then $\phi_{2j}(\epsilon)=e^{-\Psi(\omega_j,\tau_j-\epsilon,\tau_j)}\to 1$. If $\mathscr{D}_j\ne\emptyset$, by $\Psi(a,\,\tau_j-\epsilon,\,\tau_j)=\psi(a,\tau_j)\epsilon+o(\epsilon)$, Taylor expansion of $e^{-\Psi(\omega_j+\kappa_T,\tau_j-\epsilon,\tau_j)}$ yields

$$\begin{aligned}\phi_{2j}(\epsilon)&=\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}\,[1-\Psi(\omega_j+\kappa_T,\,\tau_j-\epsilon,\,\tau_j)+o(\epsilon)]\\&=-\epsilon\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}\,\psi(\omega_j+\kappa_T,\tau_j)+o(\epsilon),\end{aligned}$$

where $\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}=0$ is used in the second equality. Then,

$$
\begin{aligned}
\frac{\phi_{2j}(\epsilon)}{\epsilon} &\to -\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}\psi(\omega_j+\kappa_T,\tau_j)\\
&= -\int\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}[1-e^{-(\omega_j+\kappa_T)'s}]\,\varphi(\mathrm{d}s\,|\,\tau_j)\\
&= \int e^{-\omega_j's}\sum_{T\subset\mathscr{D}_j}(-1)^{|T|}e^{-\kappa_T's}\,\varphi(\mathrm{d}s\,|\,\tau_j)\\
&= \int e^{-\omega_j's}\prod_{i\in\mathscr{D}_j}(1-e^{-\kappa_i's})\,\varphi(\mathrm{d}s\,|\,\tau_j)\,\mathrm{d}s.
\end{aligned}
$$

Noting $\kappa_i's=s_i$, (3.3) follows by dividing (S1.2) by $\epsilon^{n_T}$ and letting $\epsilon\to 0$. □

**Proof of Proposition 3.3.** From Theorem 3.2, we have

$$
L(\gamma,h,c\,|\,D_{\mathrm{obs}})=\prod_{j=1}^{N}e^{-\phi_j(c)}\prod_{\mathscr{D}_j\neq\emptyset}\psi_j(c),
$$

where

$$
\phi_j(c)=\int_{\tau_{j-1}}^{\tau_j}\mathrm{d}t\int(1-e^{-\varrho_j's})\,c\nu(c\,\mathrm{d}s\,|\,t)=\int_{\tau_{j-1}}^{\tau_j}\mathrm{d}t\int c(1-e^{-\varrho_j's/c})\nu(\mathrm{d}s\,|\,t),
$$

and for each $j$ with $\mathscr{D}_j\neq\emptyset$,

$$
\begin{aligned}
\psi_j(c) &= \int e^{-\omega_j's}\prod_{i\in\mathscr{D}_j}(1-e^{-s_i})\,c\nu(c\mathrm{d}s\,|\,\tau_j)\\
&= \int ce^{-\omega_j's/c}\prod_{i\in\mathscr{D}_j}(1-e^{-s_i/c})\,\nu(\mathrm{d}s\,|\,\tau_j).
\end{aligned}
$$

For $c>0$ and $x>0$, $c(1-e^{-x/c})\le x$ and as $c\to\infty$, $c(1-e^{-x/c})\to x$. So by (3.4) and dominated convergence,

$$
\begin{aligned}
\sum_{j=1}^{N}\phi_j(c) &\to \sum_{j=1}^{N}\int_{\tau_{j-1}}^{\tau_j}\mathrm{d}t\int\varrho_j's\,\nu(\mathrm{d}s\,|\,t)\\
&= \sum_{j=1}^{N}\int_{\tau_{j-1}}^{\tau_j}\left[\sum_{i\in\mathscr{R}_j}\int s_i\,\nu(\mathrm{d}s\,|\,t)\right]\mathrm{d}t\\
&= \sum_{i=1}^{n}\sum_{j:\,i\in\mathscr{R}_j}\int_{\tau_{j-1}}^{\tau_j}m_i'(t)\,\mathrm{d}t=\sum_{i=1}^{n}\int_0^{y_i}m_i'(t)\,\mathrm{d}t=\sum_{i=1}^{n}m_i(y_i).
\end{aligned}
$$

On the other hand, by (3.4), with probability 1, $m'_i(\tau_j) = \int s_i\, \nu(\mathrm{d}s \,|\, \tau_j) < \infty$ for all $j \le N$. If $|\mathscr{D}_j| = 1$, then

$$\psi_j(c) \to \prod_{i \in \mathscr{D}_j} \int s_i\, \nu(\mathrm{d}s \,|\, \tau_j) = m'_i(\tau_j),$$

while if $|\mathscr{D}_j| > 1$, then by $c \prod_{i \in \mathscr{D}_j} (1 - e^{-s_i/c}) \to 0$, $\psi_j(c) \to 0$. Therefore, the limit of $\prod_{\mathscr{D}_j \ne \emptyset} \psi_j(c)$ is nonzero if and only if each nonempty $\mathscr{D}_j$ is a singleton, in which case the limit is $\prod_{\delta_i = 1} m'(y_i)$. This finishes the proof. □

**Proof of Proposition 3.4.** To prove (3.5), it suffices to consider $n = 2$. If $T_1 = T_2 = t$, then $N = n_T = 1$, $\tau_1 = t$, $\mathscr{D} = \mathscr{D}_1 = \{1, 2\}$, $\mathscr{N} = \mathscr{N}_1 = \emptyset$ and $\mathscr{R}_1 = \{1, 2\}$. Then by Theorem 3.2, it is straightforward to get the likelihood equal to $e^{-\Psi(\kappa_1+\kappa_2,t)}[\psi(\kappa_1,t)+\psi(\kappa_2,t)-\psi(\kappa_1+\kappa_2,t)]$, where $\psi$ is defined in (S1.1). By homogeneity, $\Psi(a,t) = \Psi_1(a)\Psi_2(t)$. Then the likelihood can be written as $e^{-\Psi_1(\kappa_1+\kappa_2)\Psi_2(t)}[\Psi_1(\kappa_1) + \Psi_1(\kappa_2) - \Psi_1(\kappa_1+\kappa_2)]\Psi'_2(t)$. Integrating the likelihood over $t$ from 0 to $\infty$ then yields (3.5).

Next, for $a \in \mathbb{R}^n_+$, $\Psi_1(a) = \int (1 - e^{-a's}) c\nu_0(c\mathrm{d}s) = \int (1 - e^{-a's/c})\, \nu_0(\mathrm{d}s)$. By dominated convergence, $\Psi_1(a) \to \int a's\, \nu_0(\mathrm{d}s) < \infty$ as $c \to \infty$. For $i \ne j$, this leads to $\Psi_1(\kappa_i) + \Psi_1(\kappa_j) - \Psi_1(\kappa_i + \kappa_j) \to 0$. On the other hand, $\Psi_1(\kappa_i) > 0$, otherwise $\int \mathrm{d}t \int s_i\, \nu(\mathrm{d}s \,|\, t) = 0$, contradicting the assumption that $H_i \ne 0$. As a result, $P(T_i = T_j) \to 0$, and hence the claim. □

**Proof of Proposition 4.1.** By $H \sim \mathscr{GP}(\alpha, c)$, for $t, u > 0$, $E[e^{-uH(t)}] = e^{-\alpha(t)M(u)}$, where $M(u) = \ln(1 + u/c)$. Since $G(t) = cH(\alpha^{-1}(t))$ is a pure jump process and $E[e^{-uG(t)}] = E[e^{-ucH(\alpha^{-1}(t))}] = \exp\{-\alpha(\alpha^{-1}(t))M(uc)\} = \exp\{-t\ln(1+u)\}$, $G$ is a standard Gamma process. For $i \le n$, write $\tilde{\gamma}_i = \gamma_i/c$ and $F_i(t) = 1 - \exp\{-G(t)\tilde{\gamma}_i\}$. Let $U_1, \ldots, U_n$ be i.i.d. $\sim U(0,1)$ independent of $H$. Given $H$, $S_i := F_i^*(U_i)$ are independent following distributions $F_i$, respectively, implying that $\alpha^{-1}(S_i)$ are independent and $P(\alpha^{-1}(S_i) > t \,|\, H) = 1 - F_i(\alpha(t)) = \exp\{-G(\alpha(t_i))\tilde{\gamma}_i\} = \exp\{-H(t_i)\gamma_i\} = P(T_i > t \,|\, H)$. As a result, $(T_1, \ldots, T_n) \sim (\alpha^{-1}(S_1), \ldots, \alpha^{-1}(S_n))$. By definition, $S_i = \inf\{t > 0 : \exp\{-G(t)\tilde{\gamma}_i\} \le 1 - U_i\} = \inf\{t > 0 : G(t) \ge -\ln(1 - U_i)/\tilde{\gamma}_i\} = G^*(\eta_i/\gamma_i)$, where $\eta_i = -c\ln(1 - U_i)$ are i.i.d. $\sim \mathrm{Exp}(c)$ and independent of $H$. □

**Proof of Theorem 4.2.** Since $G$ is strictly increasing, $G^*(\theta) = \inf\{t > 0 : G(t) > \theta\}$, i.e., $G^*(\theta)$ is the first passage time of $G$ across level $\theta$. For ease of notation, denote $\tau(\theta) = G^*(\theta)$ and $\zeta(\theta) = G(G^*(\theta))$ in the rest of the proof. For $t > 0$, $P(\tau(\theta) \le t) = P(G(t) \ge \theta)$. Since $G(t) \sim \text{Gamma}(t, 1)$, we see that (4.1) holds. To show (4.2), from classical results (cf. Bertoin (1996)), $P(\tau(\theta) > 0, \zeta(\theta) > \theta) = 1$ for every $\theta > 0$. Therefore, if we can show that $\tau(\theta)$ and $\zeta(\theta)$ have a joint probability density at any $t > 0$ and $s > \theta$ as follows,

$$f_\theta(t,s) = f(t,s,\theta) = \frac{e^{-s}}{\Gamma(t)} \int_0^\theta \frac{z^{t-1}\,\mathrm{d}z}{s-z}, \tag{S1.3}$$

then we obtain the conditional distribution in Theorem 4.2.

By Theorem 49.2 of Sato (1999), for any $q, u, v > 0$ with $u \ne v$,

$$\int_0^\infty e^{-u\theta} E\left[e^{-q\tau(\theta) - v(\zeta(\theta)-\theta)}\right] \mathrm{d}\theta = \frac{1}{u-v}\left[1 - \frac{\psi_q(u)}{\psi_q(v)}\right], \tag{S1.4}$$

where, denoting by $g_t$ the probability density function of $G(t)$,

$$\psi_q(u) = \exp\left\{\int_0^\infty t^{-1} e^{-qt}\,\mathrm{d}t \int_0^\infty (e^{-us} - 1) g_t(s)\,\mathrm{d}s\right\},$$

If we can show that, for $f$ in (S1.3),

$$\int_0^\infty e^{-u\theta}\left[\int_0^\infty \mathrm{d}t \int_\theta^\infty e^{-qt - v(s-\theta)} f(t,s,\theta)\,\mathrm{d}s\right] \mathrm{d}\theta = \frac{1}{u-v}\left[1 - \frac{\psi_q(u)}{\psi_q(v)}\right], \tag{S1.5}$$

then the left hand sides of (S1.4) and (S1.5) are equal. Because the Laplace transform is one-to-one, this implies the joint density of $\tau(\theta)$ and $\zeta(\theta)$ is $f_\theta(t,s) = f(t,s,\theta)$.

We evaluate the left hand side of (S1.5), denoted by $L$. Plugging (S1.3),

$$\begin{aligned} L &= \int_0^\infty e^{-u\theta}\left\{\int_0^\infty \mathrm{d}t \int_\theta^\infty e^{-qt-v(s-\theta)} \left[\frac{e^{-s}}{\Gamma(t)} \int_0^\theta \frac{z^{t-1}\,\mathrm{d}z}{s-z}\right] \mathrm{d}s\right\} \mathrm{d}\theta \\ &= \int I\{t \ge 0,\ 0 < z \le \theta \le s\}\, e^{-(u-v)\theta - qt - (v+1)s} \frac{z^{t-1}}{\Gamma(t)(s-z)}\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

Integrate over $\theta$ and then make change of variable $y = s - z$ to get

$$\begin{aligned} L &= \frac{1}{u-v}\int I\{t \ge 0,\ 0 < z \le s\}\, e^{-qt-(v+1)s} \\ &\qquad \times \left[e^{-(u-v)z} - e^{-(u-v)s}\right] \frac{z^{t-1}}{\Gamma(t)(s-z)}\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}t \\ &= \frac{1}{u-v}\int I\{t, y, z \ge 0\}\, e^{-qt-(u+1)z}\left[e^{-(v+1)y} - e^{-(u+1)y}\right] \frac{z^{t-1}}{\Gamma(t) y}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \end{aligned}$$

Now, integrate over $z > 0$ and then over $t > 0$ to get

$$\begin{aligned} L &= \frac{1}{u-v}\int I\{t, y \ge 0\}\, e^{-qt}\left[e^{-(v+1)y} - e^{-(u+1)y}\right] \frac{\mathrm{d}y\,\mathrm{d}t}{(u+1)^t y} \\ &= \frac{1}{u-v}\,\frac{1}{q + \ln(u+1)} \int_0^\infty (e^{-vy} - e^{-uy}) \frac{e^{-y}\,\mathrm{d}y}{y}. \end{aligned}$$

Then, by the Frullani integral (Bertoin (1996)), i.e., $\int_0^\infty s^{-1}(1-e^{-as})e^{-bs}\,\mathrm{d}s = \ln(1+a/b)$, $a=0$ or $b>\max(0,-a)$, we get

$$L = \frac{1}{u-v}\,\frac{\ln(u+1)-\ln(v+1)}{q+\ln(u+1)}.$$

The right hand side of (S1.5) is easier. By $g_t(s) = s^{t-1}e^{-s}/\Gamma(t)$ and the Frullani integral,

$$\begin{aligned}\psi_q(u) &= \exp\left[\int_0^\infty t^{-1}e^{-qt}\,\mathrm{d}t\int_0^\infty (e^{-us}-1)\frac{s^{t-1}e^{-s}}{\Gamma(t)}\,\mathrm{d}s\right]\\ &= \exp\left\{\int_0^\infty t^{-1}e^{-qt}\left[\frac{1}{(u+1)^t}-1\right]\,\mathrm{d}t\right\} = \frac{q}{q+\ln(u+1)}.\end{aligned}$$

Similarly, $\psi_q(v)$ can be computed. As a result, the right hand side of (S1.5) is

$$\frac{1}{u-v}\left[1-\frac{\psi_q(u)}{\psi_q(v)}\right] = \frac{1}{u-v}\left[1-\frac{q+\ln(v+1)}{q+\ln(u+1)}\right] = L,$$

finishing the proof of (4.2).

For the rest of the proof, suppose $(\tau_1, r_1) = (\tau(\theta_1), \zeta(\theta_1))$ has been sampled and there is some $s<n$ such that $\theta_s \le r_1 < \theta_{s+1}$. Then the first $s$ failure times $\tau(\theta_1), \ldots, \tau(\theta_s)$ are clearly equal to $\tau_1$. On the other hand, since $\tau(\theta_1)$ is a stopping time of $G$, the sampling of $(\tau(\theta_{s+1}), \zeta(\theta_{s+1}))$ described in the theorem is a direct consequence of the strong Markov property of $G$ (cf. Bertoin (1996)). The same argument also applies to the sampling of the other failure times and the values of $G$ at those times. □

## S2. Possible Extensions

Most attention of the paper is on homogeneous Gamma processes as models for the wear process $H$. Here we briefly discuss how its results could be extended to other pure jump processes, in particular, the Beta processes, which contain Dirichlet processes as a subclass (Lee and Kim (2004)).

A Beta process is typically specified by its corresponding cumulative hazard process

$$A(t) = \int_{[0,t]} \frac{\mathrm{d}F(t)}{1-F(t-)}, \quad \text{with } F(t) = 1-e^{-H(t)}.$$

From a practical point of view, it is reasonable to assume that the Lévy measure of $A$ varies smoothly over time, so that

$$E[e^{-\theta A(t)}] = \exp\left\{-\int_0^t \mathrm{d}v\int_0^1 (1-e^{-\theta z})\,a(z,v)\,\mathrm{d}z\right\}, \quad \theta \ge 0,$$

where $a(z,t) = \alpha(t)z^{-1}(1-z)^{\sigma(t)-1}$ with $\alpha(t) \geq 0$, $\sigma(t) > 0$, and $z \in (0,1)$. Thus, the Lévy measure of $A$ is $\varphi_A(\mathrm{d}z \,|\, t) = (1-e^{-\theta z})\, a(z,t)\, \mathrm{d}z$. Now from $\mathrm{d}H(t) = -\ln[1-\mathrm{d}A(t)]$ (Hjort (1990), p. 1274), the Lévy measure of $H$ is $\varphi_H(\mathrm{d}s \,|\, t) = f_0(s,t)\, \mathrm{d}s$, with

$$f_0(s,t) = e^{-s}a(1-e^{-s},t) = \alpha(t)e^{-\sigma(t)s}/(1-e^{-s}), \quad s > 0,\ t > 0.$$

Suppose $0 < \inf \sigma \leq \sup \sigma < \infty$ and $\sup \alpha < \infty$. Let $c = \sup \sigma$ and $F(t) = \int_0^t \alpha/c$. It is easy to check that $h(s,t) := f_0(s,t) - \alpha(t)e^{-cs}/s$ is nonnegative, bounded, and integrable. As a result, $H$ is the sum of a process following $\mathscr{G}\mathscr{P}(cF,c)$ and an independent compound Poisson process with time-dependent Lévy density $h$. Sampling for compound Poisson processes is standard (Devroye (1986)). Then sampling of survival data from $\mathcal{W} = \gamma H$ can be done by combining the DFS algorithm and the sampling for the compound Poisson process (Chi (2012)). Similar conclusion can be made under the weaker condition $0 < \inf_{t \leq b} \sigma(t) \leq \sup_{t \leq b} \sigma(t) < \infty$ and $\sup_{t \leq b} \alpha(t) < \infty$ for any finite $b > 0$, which is satisfied by Dirichlet processes as they have $\alpha(t)$ being constant and $1 - \sigma(t)$ being a cumulative distribution function on $(0, \infty)$.

On the other hand, for the above $H$, in general the joint likelihood function (3.3) for $\mathcal{W} = \gamma H$ has no semi-closed form formula. In particular, a closed form expression of the function $\Psi$ therein is unavailable. This makes it difficult to do model selection and posterior sampling of both $\beta$ and the parameters of $H$, even though the posterior sampling of $\beta$ alone can be done (Damien, Laud, and Smith (1996); Laud, Damien, and Smith (1998); Lee and Kim (2004)).

Notice that since $f_0(s,t) - \alpha(t)e^{-cs}/s = O(1)$ as $s \to 0$, the Beta and the Gamma processes are similar in terms of generating small jumps. They are significantly different only when generating large jumps. Consequently, if a data set exhibits few ties or implies the underlying wear process rarely generates large jumps, then, in order to fit the data, a Gamma process is likely to be a good substitute for a Beta process.

Finally, we briefly comment on possible extensions to the generalized Beta processes which were briefly discussed at the end of Section 3.3. In probability theory, the sampling of failure times is closely related to the sampling of the so-called first-passage events of a process. For a large class of pure jump processes with stationary increments, including the generalized Beta processes, an exact sampling method for the first-passage events has been found (Chi (2012)). It should not be difficult to extend the method to the same type processes but with nonstationary increments and to the sampling of failure times with ties. However, it is still an open question whether likelihood functions can be derived for such processes in closed or semi-closed form.

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